

6d conformal gravity

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Abstract

In the framework of ordinary-derivative approach, conformal gravity in space-time of dimension six is studied. The field content, in addition to conformal graviton field, includes two auxiliary rank-2 symmetric tensor fields, two Stueckelberg vector fields and one Stueckelberg scalar field. Gauge invariant Lagrangian with conventional kinetic terms and the corresponding gauge transformations are obtained. One of the rank-2 tensor fields and the scalar field have canonical conformal dimension. With respect to these fields, the Lagrangian contains, in addition to other terms, a cubic potential. Gauging away the Stueckelberg fields and excluding the auxiliary fields via equations of motion, the higher-derivative Lagrangian of $6d$ conformal gravity is obtained. The higher-derivative Lagrangian involves quadratic and cubic curvature terms. This higher-derivative Lagrangian coincides with the simplest Weyl invariant density discussed in the earlier literature. Generalization of de Donder gauge conditions to $6d$ conformal fields is also obtained.

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1 Introduction

In view of their aesthetic features, conformal fields have attracted considerable interest for a long period of time (see Ref.[1, 2, 3]). In space-time of dimension $d \geq 4$, conformal fields can be separated into two groups: fundamental fields and shadow fields. A field having Lorentz algebra spin s and conformal dimension $\Delta = s + d - 2$ is referred to as fundamental field, while a field having Lorentz algebra spin s and dual conformal dimension $\Delta = 2 - s$ is referred to as shadow field¹. In this paper we deal only with shadow fields which will be referred to simply as conformal fields in what follows.

The conformal fields are used, among other things, to discuss conformally invariant Lagrangians (see e.g. [1, 2, 3]). With the exception of some particular cases, Lagrangian formulation of the conformal fields involve higher derivatives and non-conventional kinetic terms. We note also that the higher derivative terms hide propagating degrees of freedom (d.o.f) of conformal fields. In Refs.[5, 6] we developed ordinary (not higher-) derivative, gauge invariant Lagrangian formulation for *free* conformal fields. That is to say that our Lagrangians for bosonic fields do not involve higher than second order terms in derivatives and have conventional kinetic terms.

In this paper we discuss $6d$ conformal gravity using the framework of ordinary-derivative approach developed in Ref.[6]. The purpose of this paper is to develop an ordinary-derivative, gauge invariant, and Lagrangian formulation for *interacting* fields of $6d$ conformal gravity². Our approach to the interacting conformal $6d$ gravity can be summarized as follows.

i) We introduce additional field degrees of freedom, i.e., we extend the space of fields entering the standard $6d$ conformal gravity. In addition to conformal graviton field, our field content involves two rank-2 symmetric tensor fields, two vector fields and one scalar field. All additional fields are supplemented by appropriate gauge symmetries³. We note that the vector fields and the scalar field turn out to be Stueckelberg fields, i.e. they are somewhat similar to the ones used in the gauge invariant approach to massive fields.

ii) Our Lagrangian for interacting fields of the $6d$ conformal gravity does not contain higher than second order terms in derivatives. To second order in fields, two-derivative contributions to the Lagrangian take the form of the standard kinetic terms of the scalar, vector, and tensor fields. Two derivative contributions appear also in the interaction vertices.

iii) Gauge transformations of fields $6d$ conformal gravity do not involve higher than first order terms in derivatives. Interacting independent one-derivative contributions to the gauge transformations take the form of the standard gauge transformations of the vector and tensor fields.

iv) The gauge symmetries of our Lagrangian make it possible to match our approach with the higher-derivative one, i.e., by an appropriate gauge fixing of the Stueckelberg fields and by solving some constraints we obtain the higher-derivative formulation of the $6d$ conformal gravity. This implies that our approach retain propagating d.o.f of the higher-derivative $6d$ conformal gravity theory, i.e., our approach is equivalent to the higher-derivative one, at least at classical level.

As is well known, the Stueckelberg approach turned out to be successful for the study of theories involving massive fields (see e.g. Ref.[9]). In fact, all covariant formulations of string theories are realized by using Stueckelberg gauge symmetries. Therefore we expect that use of the Stueck-

¹ Lorentz algebra label s is used for description of the totally symmetric fields. To discuss so called mixed-symmetry fields one needs to involve more labels of the Lorentz algebra. Discussion of conformal mixed-symmetry fields may be found in Refs.[2, 4].

² Ordinary-derivative approach to interacting $4d$ conformal gravity was discussed in Ref.[6].

³ To realize those additional gauge symmetries we adopt the approach of Refs.[5]-[8] which turns out to be the most useful for our purposes.

elberg fields for the studying conformal fields might be useful for developing new interesting formulations of the $6d$ conformal theory.

The rest of the paper is organized as follows.

Sec. 2 is devoted to the discussion of free spin-2 conformal field in $6d$ flat space. In Sec. 2.1 we start with brief review of the higher-derivative formulation of free $6d$ conformal gravity. After this, in Sec. 2.2, we review ordinary-derivative formulation of free $6d$ conformal gravity. We discuss various representation for the gauge invariant Lagrangian. We review gauge symmetries of the Lagrangian and realization of global conformal algebra symmetries on the space of gauge fields.

In Sec. 3 we describe the ordinary-derivative formulation of interacting theory of $6d$ conformal gravity. We discuss ordinary-derivative gauge invariant Lagrangian and its gauge symmetries.

In Sec. 4, we show how the higher-derivative Lagrangian of interacting $6d$ conformal gravity is obtained from our ordinary-derivative Lagrangian.

Section 5 suggests directions for future research.

In Appendix A, we summarize our conventions and the notation. In Appendix B we present some details of the derivation of gauge invariant Lagrangian and the corresponding gauge transformations.

2 Free spin-2 conformal field in $6d$ flat space

To make contact with studies in earlier literature we start with presentation of the standard, i.e. higher-derivative, formulation for the spin-2 conformal field propagating in $6d$ flat space. In the literature, such field is often referred to as conformal Weyl graviton.

2.1 Higher-derivative formulation of spin-2 conformal field

To discuss higher-derivative and gauge invariant formulation of spin-2 conformal field one uses rank-2 the Lorentz algebra $so(5, 1)$ tensor field ϕ^{ab} having conformal dimension $\Delta_{\phi^{ab}} = 0$. The field ϕ^{ab} is symmetric, $\phi^{ab} = \phi^{ba}$, and traceful $\phi^{aa} \neq 0$. Higher-derivative Lagrangian for the field ϕ^{ab} is given by

$$\mathcal{L} = \frac{1}{3} C_{\text{lin}}^{abce} \square C_{\text{lin}}^{abce}, \quad (2.1)$$

where C_{lin}^{abce} is the linearized Weyl tensor. Using representation of the Weyl tensor in terms of curvatures

$$\begin{aligned} C^{abce} &= R^{abce} - \frac{1}{4}(\eta^{ac} R^{be} - \eta^{bc} R^{ae} + \eta^{be} R^{ac} - \eta^{ae} R^{bc}) \\ &+ \frac{1}{20}(\eta^{ac} \eta^{be} - \eta^{ae} \eta^{bc}) R, \end{aligned} \quad (2.2)$$

and the Gauss-Bonnet relation

$$R_{\text{lin}}^{abce} \square R_{\text{lin}}^{abce} - 4 R_{\text{lin}}^{ab} \square R_{\text{lin}}^{ab} + R_{\text{lin}} \square R_{\text{lin}} = 0 \quad (\text{up to total derivative}), \quad (2.3)$$

we obtain the representation for Lagrangian (2.1) in terms of linearized Ricci curvatures,

$$\mathcal{L} = R_{\text{lin}}^{ab} \square R_{\text{lin}}^{ab} - \frac{3}{10} R_{\text{lin}}^2, \quad (2.4)$$

which is also useful for certain purposes. Using explicit representation of the Ricci curvatures in terms of the field ϕ^{ab}

$$R_{\text{lin}}^{ab} = \frac{1}{2} \left(-\square \phi^{ab} + \partial^a \partial^c \phi^{bc} + \partial^b \partial^c \phi^{ac} - \partial^a \partial^b \phi^{cc} \right), \quad (2.5)$$

$$R_{\text{lin}} = \partial^a \partial^b \phi^{ab} - \square \phi^{aa}, \quad (2.6)$$

leads to other well known form of the Lagrangian

$$\mathcal{L} = \frac{1}{4} \phi^{ab} \square^3 P^{abce} \phi^{ce}, \quad (2.7)$$

where we use the notation as in [1]:

$$P^{abce} \equiv \frac{1}{2} (\pi^{ac} \pi^{be} + \pi^{ae} \pi^{bc}) - \frac{1}{5} \pi^{ab} \pi^{ce}, \quad \pi^{ab} \equiv \eta^{ab} - \frac{\partial^a \partial^b}{\square}. \quad (2.8)$$

Lagrangian (2.1) is invariant under linearized diffeomorphism and Weyl gauge transformations

$$\delta \phi^{ab} = \partial^a \xi^b + \partial^b \xi^a + \eta^{ab} \xi, \quad (2.9)$$

where ξ^a and ξ are the respective diffeomorphism and Weyl gauge transformation parameters.

We now discuss on-shell d.o.f of 6d conformal gravity. To this end we use fields transforming in irreps of $so(4)$ algebra. One can prove that on-shell d.o.f are described by three rank-2 *traceless* symmetric tensor fields $\phi_{k'}^{ij}$, two vector fields $\phi_{k'}^i$, and one scalar field ϕ_0 :⁴

$$\begin{array}{ccc} \phi_{-2}^{ij} & \phi_0^{ij} & \phi_2^{ij} \\ \phi_{-1}^i & \phi_1^i & \\ \phi_0 & & \end{array} \quad (2.10)$$

$i, j = 1, \dots, 4$ (for details see Appendix B in Ref.[6]). Total number of on-shell d.o.f shown in (2.10) is given by

$$\mathbf{n} = 36. \quad (2.11)$$

We note that this \mathbf{n} is decomposed in a sum of d.o.f for fields given in (2.10) as⁵:

$$\mathbf{n} = \sum_{k'=0,\pm 2} \mathbf{n}(\phi_{k'}^{ij}) + \sum_{k'=\pm 1} \mathbf{n}(\phi_{k'}^i) + \mathbf{n}(\phi_0), \quad (2.12)$$

$$\mathbf{n}(\phi_{k'}^{ij}) = 9, \quad k' = 0, \pm 2; \quad (2.13)$$

$$\mathbf{n}(\phi_{k'}^i) = 4, \quad k' = \pm 1; \quad (2.14)$$

$$\mathbf{n}(\phi_0) = 1. \quad (2.15)$$

⁴ Fields (2.10) are related to non-unitary representation of conformal algebra $so(6, 2)$. Discussion of unitary representations of the conformal algebra that are relevant for elementary particles may be found e.g. in Refs.[10],[11].

⁵ Total d.o.f given in (2.11) was found in Ref.[1]. Decomposition of \mathbf{n} (2.12) into irreps of the $so(4)$ algebra was carried out in Ref.[6] (see Appendix B in Ref.[6]). Light-cone gauge approach used in Ref.[6] provides easy possibility to decompose the total \mathbf{n} into irreps of $so(4)$ algebra. Discussion of other methods for counting propagating d.o.f of higher-derivative theories may be found in Refs.[12, 13].

2.2 Ordinary-derivative formulation of spin-2 conformal field

We now review the ordinary-derivative formulation of the spin-2 conformal field in $6d$ flat space developed in Ref.[6]. In addition to results in Ref.[6], we discuss also two new representations for gauge invariant Lagrangian. One of the new representations turns out to be convenient for the generalization to theory of interacting spin-2 conformal field. Also we present our results for de Donder like gauge conditions which have not been discussed in the earlier literature.

Field content. To discuss ordinary-derivative and gauge invariant formulation of the spin-2 conformal field in $6d$ flat space we use three rank-2 tensor fields $\phi_{k'}^{ab}$, two vector fields $\phi_{k'}^a$, and one scalar field ϕ_0 :

$$\begin{array}{ccc} \phi_{-2}^{ab} & \phi_0^{ab} & \phi_2^{ab} \\ \phi_{-1}^a & \phi_1^a & \\ & \phi_0 & \end{array} \quad (2.16)$$

The fields $\phi_{k'}^{ab}$, $\phi_{k'}^a$ and ϕ_0 are the respective rank-2 tensor, vector, and scalar fields of the Lorentz algebra $so(5, 1)$. Note that the tensor fields $\phi_{k'}^{ab}$ are symmetric, $\phi_{k'}^{ab} = \phi_{k'}^{ba}$, and traceful, $\phi_{k'}^{aa} \neq 0$. Fields in (2.16) have the conformal dimensions

$$\begin{aligned} \Delta_{\phi_{k'}^{ab}} &= 2 + k', & k' &= 0, \pm 2, \\ \Delta_{\phi_{k'}^a} &= 2 + k', & k' &= \pm 1, \\ \Delta_{\phi_0} &= 2. \end{aligned} \quad (2.17)$$

Gauge invariant Lagrangian. We discuss three representations for Lagrangian in turn.

1st representation for the Lagrangian. This representation found in Ref.[6] is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \phi_2^{ab} (E_{EH} \phi_{-2})^{ab} + \frac{1}{4} \phi_0^{ab} (E_{EH} \phi_0)^{ab} + \phi_1^a (E_{Max} \phi_{-1})^a + \frac{1}{2} \phi_0 \square \phi_0 \\ &+ \phi_1^a \partial^b \chi_0^{ab} + \phi_{-1}^a \partial^b \chi_2^{ba} - \frac{1}{2} \chi_0^{ab} \phi_2^{ab}, \end{aligned} \quad (2.18)$$

$$\chi_0^{ab} \equiv \phi_0^{ab} - \eta^{ab} \phi_0^{cc} - u \eta^{ab} \phi_0, \quad (2.19)$$

$$\chi_2^{ab} \equiv \phi_2^{ab} - \eta^{ab} \phi_2^{cc}, \quad (2.20)$$

$$u \equiv \sqrt{5/2}, \quad (2.21)$$

where E_{EH} and E_{Max} are the respective second-order Einstein-Hilbert and Maxwell operators,

$$(E_{EH} \phi)^{ab} = \square \phi^{ab} - \partial^a \partial^c \phi^{cb} - \partial^b \partial^c \phi^{ca} + \partial^a \partial^b \phi^{cc} + \eta^{ab} (\partial^c \partial^e \phi^{ce} - \square \phi^{cc}), \quad (2.22)$$

$$(E_{Max} \phi)^a = \square \phi^a - \partial^a \partial^b \phi^b. \quad (2.23)$$

Thus, we see that two-derivative contributions to Lagrangian (2.18) takes the form of standard second-order kinetic terms for the respective rank-2 tensor fields, vector fields and scalar field. Note also that besides the two-derivative contributions, the Lagrangian involves one-derivative contributions and derivative-independent mass-like contributions.

2nd representation for the Lagrangian. The second representation has not been discussed in the earlier literature. For the reader convenience, we discuss this representation because it allows us to introduce de Donder like gauge conditions for $6d$ conformal gravity⁶. This is to say that Lagrangian (2.18) can be represented as (up to total derivative)

$$\begin{aligned}\mathcal{L} = & \frac{1}{2}\phi_2^{ab}\Box\phi_{-2}^{ab} - \frac{1}{4}\phi_2^{aa}\Box\phi_{-2}^{bb} + \frac{1}{4}\phi_0^{ab}\Box\phi_0^{ab} - \frac{1}{8}\phi_0^{aa}\Box\phi_0^{bb} + \phi_1^a\Box\phi_{-1}^a + \frac{1}{2}\phi_0\Box\phi_0 \\ & + C_{-1}^a C_3^a + \frac{1}{2}C_1^a C_1^a + C_0 C_2 \\ & - \frac{1}{2}\phi_2^{ab}\phi_0^{ab} + \frac{1}{4}\phi_2^{aa}\phi_0^{bb} - \frac{1}{2}\phi_1^a\phi_1^a,\end{aligned}\tag{2.24}$$

where quantities $C_{k'}^a$, $C_{k'}$, which we refer to as conformal de Donder divergences, are given by

$$C_{-1}^a = \partial^b\phi_{-2}^{ab} - \frac{1}{2}\partial^a\phi_{-2}^{bb} + \phi_{-1}^a,\tag{2.25}$$

$$C_1^a = \partial^b\phi_0^{ab} - \frac{1}{2}\partial^a\phi_0^{bb} + \phi_1^a,\tag{2.26}$$

$$C_3^a = \partial^b\phi_2^{ab} - \frac{1}{2}\partial^a\phi_2^{bb},\tag{2.27}$$

$$C_0 = \partial^a\phi_{-1}^a + \frac{1}{2}\phi_0^{aa} + u\phi_0,\tag{2.28}$$

$$C_2 = \partial^a\phi_1^a + \frac{1}{2}\phi_2^{aa}.\tag{2.29}$$

We note that it is the conformal de Donder divergencies that define de Donder like gauge conditions for our conformal $6d$ fields,

$$\begin{aligned}C_{k'}^a &= 0, & k' &= -1, 1, 3, \\ C_{k'} &= 0, & k' &= 0, 2.\end{aligned}\tag{2.30}$$

conformal de Donder gauge conditions.

The fields described by Lagrangian (2.24) are related to non-unitary representation of the conformal algebra. Fields with $k' = 0$ have kinetic terms with correct signs. The remaining fields with $k' \neq 0$ can be collected into the vector and tensor doublets, ϕ_{-1}^a, ϕ_1^a and $\phi_{-2}^{ab}, \phi_2^{ab}$. We note then that vector (tensor) doublet describes one vector (tensor) field with wrong sign of kinetic term and one vector (tensor) field with correct sign of the kinetic term.

3rd representation for the Lagrangian. Finally, we discuss representation for free Lagrangian (2.18) which turns to be most adapted for generalization to interacting $6d$ conformal gravity. This is to say that Lagrangian (2.18) can be represented as (up to total derivative)

$$\mathcal{L} = \sum_{a=1}^6 \mathcal{L}_a,\tag{2.31}$$

⁶ De Donder gauges turn out to be useful for study of various dynamical systems. Recent discussion of the *standard* de Donder-Feynman gauge for massless fields may be found in Refs.[14, 15, 16]. Applications of modified de Donder gauges for massless and massive *AdS* fields [17] to studying the *AdS/CFT* correspondence may found in Ref.[18].

$$\mathcal{L}_1 = -\phi_2^{ab} \widehat{G}_{\text{lin}}^{(ab)}, \quad (2.32)$$

$$\mathcal{L}_2 = -\frac{1}{4} \partial^c \phi_0^{ab} \partial^c \phi_0^{ab} + \frac{1}{8} \partial^c \phi_0^{aa} \partial^c \phi_0^{bb} + \frac{1}{2} C_1^a C_1^a, \quad (2.33)$$

$$\mathcal{L}_3 = -\frac{1}{2} F^{ab}(\phi_{-1}) F^{ab}(\phi_1), \quad (2.34)$$

$$\mathcal{L}_4 = -\frac{1}{2} \partial^a \phi_0 \partial^a \phi_0, \quad (2.35)$$

$$\mathcal{L}_5 = \phi_1^a \partial^b \chi_0^{ab}, \quad (2.36)$$

$$\mathcal{L}_6 = -\frac{1}{2} \phi_2^{ab} \chi_0^{ab}, \quad (2.37)$$

$$\widehat{G}_{\text{lin}}^{(ab)} = G_{\text{lin}}^{ab} + \frac{1}{2} (\partial^a \phi_{-1}^b + \partial^b \phi_{-1}^a) - \eta^{ab} \partial^c \phi_{-1}^c, \quad (2.38)$$

$$G_{\text{lin}}^{ab} = R_{\text{lin}}^{ab}(\phi_{-2}) - \frac{1}{2} \eta^{ab} R_{\text{lin}}(\phi_{-2}), \quad (2.39)$$

$$C_1^a \equiv \partial^b \phi_0^{ab} - \frac{1}{2} \partial^a \phi_0^{bb}, \quad (2.40)$$

$$F^{ab}(\phi_{k'}) \equiv \partial^a \phi_{k'}^b - \partial^b \phi_{k'}^a, \quad k' = \pm 1, \quad (2.41)$$

where χ_0^{ab} is defined in (2.19). The linearized Ricci curvatures for the field ϕ_{-2}^{ab} in (2.39) are obtained by substituting the field ϕ_{-2}^{ab} in the respective expressions (2.5) and (2.6). Note that linearized Einstein tensor G_{lin}^{ab} (2.39) can be represented by using operator E_{EH} (2.22) as

$$G_{\text{lin}}^{ab} = -\frac{1}{2} (E_{EH} \phi_{-2})^{ab}. \quad (2.42)$$

Also, we note that shifted linearized Einstein tensor \widehat{G}^{ab} (2.38) can be expressed in terms of the corresponding shifted linearized Ricci curvatures

$$\widehat{G}_{\text{lin}}^{ab} = \widehat{R}_{\text{lin}}^{ab} - \frac{1}{2} \eta^{ab} \widehat{R}_{\text{lin}}, \quad (2.43)$$

where the shifted linearized curvatures are defined by relations

$$\widehat{R}_{\text{lin}}^{abce} = R_{\text{lin}}^{abce} + \eta^{ac} \varphi_{\text{lin}}^{be} - \eta^{bc} \varphi_{\text{lin}}^{ae} + \eta^{be} \varphi_{\text{lin}}^{ac} - \eta^{ae} \varphi_{\text{lin}}^{bc}, \quad (2.44)$$

$$R_{\text{lin}}^{abce} = \frac{1}{2} (-\partial^a \partial^c \phi_{-2}^{be} + \partial^b \partial^c \phi_{-2}^{ae} - \partial^b \partial^e \phi_{-2}^{ac} + \partial^a \partial^e \phi_{-2}^{bc}), \quad (2.45)$$

$$\widehat{R}_{\text{lin}}^{ab} = R_{\text{lin}}^{ab} + 4\varphi_{\text{lin}}^{ab} + \eta^{ab} \varphi_{\text{lin}}^{cc}, \quad (2.46)$$

$$\widehat{R}_{\text{lin}} = R_{\text{lin}} + 10\varphi_{\text{lin}}^{cc}, \quad (2.47)$$

$$\varphi_{\text{lin}}^{ab} = q \partial^a \phi_{-1}^b, \quad q = \frac{1}{4}, \quad (2.48)$$

$$\widehat{R}^{ab} = \widehat{R}^{cacb}, \quad \widehat{R} = \widehat{R}^{aa}. \quad (2.49)$$

Gauge transformations. We now discuss gauge symmetries of Lagrangian (2.18). To this end we introduce the gauge transformation parameters,

$$\begin{array}{ccc} \xi_{-3}^a & \xi_{-1}^a & \xi_1^a \\ \xi_{-2} & \xi_0 & \end{array} \quad (2.50)$$

Conformal dimensions of the gauge transformation parameters are given by

$$\begin{aligned} \Delta_{\xi_{k'}^a} &= 2 + k', & k' &= -3, -1, 1, \\ \Delta_{\xi_{k'}} &= 2 + k', & k' &= -2, 0. \end{aligned} \quad (2.51)$$

The gauge transformation parameters $\xi_{k'}^a$ and $\xi_{k'}$ are the respective vector and scalar fields of the Lorentz algebra $so(5, 1)$. The Lagrangian is invariant under the gauge transformations

$$\delta\phi_{-2}^{ab} = \partial^a \xi_{-3}^b + \partial^b \xi_{-3}^a + \frac{1}{2} \eta^{ab} \xi_{-2}, \quad (2.52)$$

$$\delta\phi_0^{ab} = \partial^a \xi_{-1}^b + \partial^b \xi_{-1}^a + \frac{1}{2} \eta^{ab} \xi_0, \quad (2.53)$$

$$\delta\phi_2^{ab} = \partial^a \xi_1^b + \partial^b \xi_1^a, \quad (2.54)$$

$$\delta\phi_{-1}^a = \partial^a \xi_{-2} - \xi_{-1}^a, \quad (2.55)$$

$$\delta\phi_1^a = \partial^a \xi_0 - \xi_1^a, \quad (2.56)$$

$$\delta\phi_0 = -u \xi_0, \quad (2.57)$$

where u is given in (2.21).

Realization of conformal algebra symmetries. In $6d$ space-time, the conformal algebra $so(6, 2)$ referred to the basis of Lorentz algebra $so(5, 1)$ consists of translation generators P^a , conformal boost generators K^a , dilatation generator D and generators of the Lorentz algebra $so(5, 1)$ denoted by J^{ab} . We assume the following normalization for commutators of the conformal algebra⁷:

$$\begin{aligned} [D, P^a] &= -P^a, & [P^a, J^{bc}] &= \eta^{ab} P^c - \eta^{ac} P^b, \\ [D, K^a] &= K^a, & [K^a, J^{bc}] &= \eta^{ab} K^c - \eta^{ac} K^b, \\ [P^a, K^b] &= \eta^{ab} D - J^{ab}, \\ [J^{ab}, J^{ce}] &= \eta^{bc} J^{ae} + 3 \text{ terms}. \end{aligned} \quad (2.58)$$

Let ϕ denotes field propagating in the flat space-time. Let Lagrangian for the free field ϕ be conformal invariant. This implies, that Lagrangian is invariant with respect to transformation (invariance of the Lagrangian is assumed to be up to total derivative)

$$\delta_{\hat{G}} \phi = \hat{G} \phi, \quad (2.59)$$

⁷ Note that in our approach only $so(5, 1)$ symmetries are realized manifestly. The $so(6, 2)$ symmetries could be realized manifestly by using ambient space approaches (see e.g. [19, 20, 21])

where a realization of the conformal algebra generators \hat{G} in terms of differential operators acting on ϕ takes the form

$$P^a = \partial^a, \quad (2.60)$$

$$J^{ab} = x^a \partial^b - x^b \partial^a + M^{ab}, \quad (2.61)$$

$$D = x^a \partial^a + \Delta, \quad (2.62)$$

$$K^a = K_{\Delta, M}^a + R^a, \quad (2.63)$$

$$K_{\Delta, M}^a \equiv -\frac{1}{2} x^b x^b \partial^a + x^a D + M^{ab} x^b. \quad (2.64)$$

In (2.61)-(2.63), Δ is operator of conformal dimension, M^{ab} is spin operator of the Lorentz algebra. Action of M^{ab} on fields of the Lorentz algebra is well known and for rank-2 tensor, vector, and scalar fields considered in this paper is given by

$$\begin{aligned} M^{ab} \phi^{ce} &= \eta^{ae} \phi^{cb} + \eta^{ac} \phi^{be} - (a \leftrightarrow b), \\ M^{ab} \phi^c &= \eta^{ac} \phi^b - (a \leftrightarrow b), \\ M^{ab} \phi &= 0. \end{aligned} \quad (2.65)$$

Relation (2.63) implies that conformal boost transformations can be presented as

$$\delta_{K^a} \phi = \delta_{K_{\Delta, M}^a} \phi + \delta_{R^a} \phi. \quad (2.66)$$

Explicit representation for the action of operator $K_{\Delta, M}^a$ (2.64) is easily obtained from the relations above-given. This is to say that the rank-2 tensor, vector, and scalar fields considered in this paper transform as

$$\begin{aligned} \delta_{K_{\Delta, M}^a} \phi_{k'}^{bc} &= K_{\Delta(\phi_{k'})}^a \phi_{k'}^{bc} + M^{abf} \phi_{k'}^{fc} + M^{acf} \phi_{k'}^{bf}, & k' = 0, \pm 2, \\ \delta_{K_{\Delta, M}^a} \phi_{k'}^b &= K_{\Delta(\phi_{k'})}^a \phi_{k'}^b + M^{abf} \phi_{k'}^f, & k' = \pm 1, \\ \delta_{K_{\Delta, M}^a} \phi_0 &= K_{\Delta(\phi_0)}^a \phi_0, \end{aligned} \quad (2.67)$$

$$K_{\Delta}^a \equiv -\frac{1}{2} x^b x^b \partial^a + x^a (x \partial + \Delta), \quad (2.68)$$

$$M^{abc} \equiv \eta^{ab} x^c - \eta^{ac} x^b. \quad (2.69)$$

Thus, all that remains is to find explicit representation for operator R^a in (2.63). The operator R^a depends on the derivative ∂^a and does not depend on the space-time coordinates x^a , $[P^a, R^b] = 0$. In the standard formulation of the conformal fields, the operator R^a is equal to zero, while in the ordinary-derivative approach, we discuss in this paper, the operator R^a is non-trivial. This implies that, in the framework of ordinary-derivative approach, the complete description of the conformal fields requires finding not only gauge invariant Lagrangian but also the operator R^a . Realization of the operator R^a on a space of gauge fields (2.16) is given by

$$\delta_{R^a} \phi_{-2}^{bc} = 0, \quad (2.70)$$

$$\delta_{R^a} \phi_0^{bc} = -2(\eta^{ab} \phi_{-1}^c + \eta^{ac} \phi_{-1}^b) + \eta^{bc} \phi_{-1}^a - 4\partial^a \phi_{-2}^{bc}, \quad (2.71)$$

$$\delta_{R^a} \phi_2^{bc} = -4(\eta^{ab} \phi_1^c + \eta^{ac} \phi_1^b) + 2\eta^{bc} \phi_1^a - 4\partial^a \phi_0^{bc}, \quad (2.72)$$

$$\delta_{R^a} \phi_{-1}^b = 4\phi_{-2}^{ab}, \quad (2.73)$$

$$\delta_{R^a} \phi_1^b = 2\phi_0^{ab} - 2u\eta^{ab} \phi_0 - 2\partial^a \phi_{-1}^b, \quad (2.74)$$

$$\delta_{R^a} \phi_0 = 2u\phi_{-1}^a. \quad (2.75)$$

From (2.70)-(2.75), we see the operator R^a maps the gauge field with conformal dimension Δ into the ones having conformal dimension less than Δ . This is to say that the realization of the operator R^a given in (2.70)-(2.75) can schematically be represented as⁸

$$\phi_2^{ab} \xrightarrow{R} \phi_1^a \oplus \partial \phi_0^{ab}, \quad \phi_0^{ab} \xrightarrow{R} \phi_{-1}^a \oplus \partial \phi_{-2}^{ab}, \quad \phi_{-2}^{ab} \xrightarrow{R} 0, \quad (2.76)$$

$$\phi_1^a \xrightarrow{R} \phi_0^{ab} \oplus \phi_0 \oplus \partial \phi_{-1}^a, \quad \phi_0 \xrightarrow{R} \phi_{-1}^a, \quad \phi_{-1}^a \xrightarrow{R} \phi_{-2}^{ab}. \quad (2.77)$$

Interrelation of the ordinary-derivative and the higher-derivative approaches. From (2.55)-(2.57), we see that both vector fields ϕ_{-1}^a , ϕ_1^a and the scalar field ϕ_0 transforms as Stueckelberg (Goldstone) fields under the gauge transformations, i.e. these fields can be gauged away by using the gauge symmetries. Gauging away the vector fields and the scalar field,

$$\phi_{\pm 1}^a = 0, \quad \phi_0 = 0, \quad (2.78)$$

we see that our Lagrangian (2.31) takes the simplified form

$$\mathcal{L} = -\phi_2^{ab} G_{\text{lin}}^{ab} - \frac{1}{4} \partial^c \phi_0^{ab} \partial^c \phi_0^{ab} + \frac{1}{8} \partial^c \phi_0^{aa} \partial^c \phi_0^{bb} + \frac{1}{2} C_1^a C_1^a - \frac{1}{2} \phi_2^{ab} \chi_0^{ab}. \quad (2.79)$$

Now using equations of motion for the rank-2 tensor field ϕ_2^{ab} obtained from Lagrangian (2.79) we find the equation

$$\phi_0^{ab} - \eta^{ab} \phi_0^{cc} = -2G_{\text{lin}}^{ab}, \quad (2.80)$$

which has the obvious solution

$$\bar{\phi}_0^{ab} = -2R_{\text{lin}}^{ab} + \frac{1}{5} \eta^{ab} R_{\text{lin}}, \quad (2.81)$$

where the linearized Ricci curvatures are obtained by substituting the field ϕ_{-2}^{ab} in (2.5),(2.6). Plugging solution $\bar{\phi}_0^{ab}$ (2.81) into Lagrangian (2.79) we obtain the higher-derivative Lagrangian given in (2.4). Thus we see that our ordinary-derivative approach is equivalent to the standard one and our field ϕ_{-2}^{ab} is identified with excitation of the conformal graviton field ϕ^{ab} in Sec. 2.1.

3 Interacting $6d$ conformal gravity

We begin our discussion of interacting theory of $6d$ conformal gravity with the description of a field content. Field content of the interacting theory is simply obtained by promoting the Minkowski

⁸ Realization of the operator R^a on space of on-shell fields can be obtained by using group theoretical methods, while the realization of R^a on space of gauge fields requires the use of the gauge invariant approach.

space free fields (2.16) to the fields in curved space-time described by metric tensor field $g_{\mu\nu}$. As usually, this metric tensor field is considered to be conformal graviton field. As we have already said, the field ϕ_{-2}^{ab} describes excitation of the conformal graviton, i.e., in the interacting theory, the field ϕ_{-2}^{ab} is related to the metric tensor field $g_{\mu\nu}$. Also note that, instead of metric-like approach to conformal gravity, we prefer to use frame-like approach, i.e., we use vielbein field e_μ^a , $g_{\mu\nu} = e_\mu^a e_\nu^a$ and fields carrying tangent-flat indices, ϕ^a , ϕ^{ab} , which are related to fields carrying base manifold indices ϕ^μ , $\phi^{\mu\nu}$, by the standard relations $\phi^a = e_\mu^a \phi^\mu$, $\phi^{ab} = e_\mu^a e_\nu^b \phi^{\mu\nu}$ (for details of our notation see Appendix A). Also, following commonly used nomenclature, we use notation b^a in place of the field ϕ_{-1}^a . To summarize, the field content we use to develop the ordinary-derivative approach to the interacting 6d conformal gravity is given by⁹

$$\begin{array}{ccc} e_\mu^a & \phi_0^{ab} & \phi_2^{ab} \\ b^a & \phi_1^a & \\ & \phi_0 & \end{array} \quad (3.1)$$

For field ϕ having Weyl dimension Δ_ϕ^w , we define local Weyl transformations in the usual way,

$$\delta\phi = \Delta_\phi^w \sigma \phi, \quad (3.2)$$

where σ is Weyl gauge transformation parameter. Using this convention, the Weyl dimensions of the fields are given by¹⁰,

$$\begin{aligned} \Delta_{e_\mu^a}^w &= -1, & \Delta_{\phi_{k'}^{ab}}^w &= 2 + k', & k' &= 0, 2, \\ \Delta_{\phi_1^a}^w &= 3, & \Delta_{\phi_0}^w &= 2. \end{aligned} \quad (3.3)$$

Gauge transformation of the compensator field b^a involves gradient term (see below), but for constant σ the field b^a transforms as in (3.2) with $\Delta_{b^a}^w = 1$. With this convention for the Weyl dimension of the field b^a , we note that conformal dimensions of fields not carrying base manifold indices, ϕ_0^{ab} , ϕ_2^{ab} , b^a , ϕ_1^a , ϕ_0 , (2.17) are equal to their Weyl dimensions (3.3).

We now discuss gauge invariant Lagrangian for interacting fields (3.1). The Lagrangian we find is given by

$$\mathcal{L} = \sum_{a=1}^8 \mathcal{L}_a, \quad (3.4)$$

$$e^{-1} \mathcal{L}_1 = -\phi_2^{ab} \widehat{G}^{(ab)}, \quad (3.5)$$

$$\begin{aligned} e^{-1} \mathcal{L}_2 &= -\frac{1}{4} \mathcal{D}^a \phi_0^{bc} \mathcal{D}^a \phi_0^{bc} + \frac{1}{8} \mathcal{D}^a \phi_0^{bb} \mathcal{D}^a \phi_0^{cc} + \frac{1}{2} C_1^a C_1^a \\ &\quad - \frac{1}{2} \widehat{R}^{cabe} \phi_0^{ab} \phi_0^{ce} + \frac{1}{2} \widehat{R}^{ab} \phi_0^{ac} \phi_0^{cb} - \frac{1}{2} \widehat{R}^{ab} \phi_0^{ab} \phi_0^{cc} + \left(\frac{1}{8} \phi_0^{aa} \phi_0^{bb} - \frac{1}{4} \phi_0^{ab} \phi_0^{ab} \right) \widehat{R}, \end{aligned} \quad (3.6)$$

⁹ The symmetric and antisymmetric parts of the gauge field associated with the conformal boosts are related to the field ϕ_0^{ab} and the field strength $F^{ab}(b)$ (see (3.24)) respectively. Also, we note that the parameter ξ_{-1}^a (see (3.29)) is related to conformal boosts gauge transformation parameter.

¹⁰ In Ref.[1], conformal dimension is referred to as canonical dimension.

$$e^{-1}\mathcal{L}_3 = -\frac{1}{2}\mathcal{F}^{ab}(b)\mathcal{F}^{ab}(\phi_1), \quad (3.7)$$

$$e^{-1}\mathcal{L}_4 = -\frac{1}{2}\mathcal{D}^a\phi_0\mathcal{D}^a\phi_0, \quad (3.8)$$

$$e^{-1}\mathcal{L}_5 = \phi_1^a\mathcal{D}^b\chi_0^{ab}, \quad (3.9)$$

$$e^{-1}\mathcal{L}_6 = -\frac{1}{2}\phi_2^{ab}\chi_0^{ab}, \quad (3.10)$$

$$e^{-1}\mathcal{L}_7 = \frac{1}{4}\phi_0^{ab}T^{ab} - \frac{u}{8}\phi_0 F^{ab}F^{ab}, \quad (3.11)$$

$$e^{-1}\mathcal{L}_8 = \frac{1}{4}\phi_0^{ab}\phi_0^{bc}\phi_0^{ca} - \frac{5}{16}\phi_0^{ab}\phi_0^{ab}\phi_0^{cc} + \frac{1}{16}(\phi_0^{aa})^3 \\ - \frac{u}{8}\phi_0^{ab}\phi_0^{ab}\phi_0 - \frac{3}{16}\phi_0^{aa}\phi_0^2 - \frac{3}{16u}\phi_0^3, \quad (3.12)$$

$$e \equiv \det e_\mu^a, \quad (3.13)$$

$$\widehat{G}^{(ab)} \equiv G^{ab} + \frac{1}{2}(D^ab + D^bb^a) + \frac{1}{4}b^ab^b - \eta^{ab}(D^cb^c - \frac{3}{8}b^cb^c), \quad (3.14)$$

$$G^{ab} \equiv R^{ab} - \frac{1}{2}\eta^{ab}R, \quad (3.15)$$

$$C_1^a \equiv \mathcal{D}^b\phi_0^{ab} - \frac{1}{2}\mathcal{D}^a\phi_0^{bb}, \quad (3.16)$$

$$\chi_0^{ab} \equiv \phi_0^{ab} - \eta^{ab}\phi_0^{cc} - u\eta^{ab}\phi_0, \quad (3.17)$$

$$T^{ab} \equiv F^{ac}F^{bc} - \frac{1}{4}\eta^{ab}F^{ce}F^{ce}, \quad (3.18)$$

$$F^{ab} \equiv D^ab^b - D^bb^a, \quad (3.19)$$

where u is defined in (2.21). Complete description of our notation may be found in Appendix A. Here we mention the most important notation.

a) For rank- s field $\phi^{b_1\dots b_s}$ having Weyl dimension Δ_ϕ^w , covariant derivative \mathcal{D}^a is defined to be

$$\mathcal{D}^a\phi^{b_1\dots b_s} = \widehat{D}^a\phi^{b_1\dots b_s} + \Delta_\phi^w q b^a\phi^{b_1\dots b_s}, \quad q = \frac{1}{4}, \quad (3.20)$$

where \widehat{D}^a is covariant derivative with shifted Lorentz connection $\widehat{\omega}_\mu^{ab}$,

$$\widehat{D}^a\phi^b = e^{\mu a}\partial_\mu\phi^b + \widehat{\omega}^{abc}\phi^c, \quad \widehat{\omega}^{abc} = e^{\mu a}\widehat{\omega}_\mu^{bc}, \quad (3.21)$$

$$D^a\phi^b = e^{\mu a}\partial_\mu\phi^b + \omega^{abc}\phi^c, \quad \omega^{abc} = e^{\mu a}\omega_\mu^{bc}, \quad (3.22)$$

$$\widehat{\omega}^{abc} = \omega^{abc} + q(\eta^{ac}b^b - \eta^{ab}b^c), \quad (3.23)$$

while D^a is a covariant derivative with the standard Lorentz connection $\omega_\mu^{bc}(e)$.

b) Field strength $\mathcal{F}^{ab}(\phi)$ for vector field ϕ^a is defined to be

$$\mathcal{F}^{ab}(\phi) \equiv \mathcal{D}^a \phi^b - \mathcal{D}^b \phi^a. \quad (3.24)$$

Note that for the compensator field b^a the field strength \mathcal{F}^{ab} becomes the standard one (3.19),

$$\mathcal{F}^{ab}(b) = F^{ab}. \quad (3.25)$$

c) Curvature \hat{R}^{abce} is defined for the shifted connection $\hat{\omega}_\mu^{ab}$ as

$$\hat{R}_{\mu\nu}{}^{ab} = \partial_\mu \hat{\omega}_\nu^{ab} - \partial_\nu \hat{\omega}_\mu^{ab} + \hat{\omega}_\mu^{ac} \hat{\omega}_\nu^{cb} - \hat{\omega}_\nu^{ac} \hat{\omega}_\mu^{cb}, \quad (3.26)$$

$$\hat{R}^{abce} = e^{\mu a} e^{\nu b} \hat{R}_{\mu\nu}{}^{ce}. \quad (3.27)$$

Ricci curvatures R^{ab} , R are defined as $R^{ab} = R^{cacb}$, $R = R^{aa}$, where $R^{abce} = e^{\mu a} e^{\nu b} R_{\mu\nu}{}^{ce}$ and $R_{\mu\nu}{}^{ab}$ is curvature for the standard Lorentz connection $\omega_\mu^{ab}(e)$. We note that the shifted Einstein tensor is defined in a usual way

$$\hat{G}^{ab} = \hat{R}^{ab} - \frac{1}{2} \eta^{ab} \hat{R}, \quad \hat{R}^{ab} = \hat{R}^{cacb}, \quad \hat{R} = \hat{R}^{aa}. \quad (3.28)$$

A few remarks are in order.

i) Contributions to interacting Lagrangian (3.4) denoted by $\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_5$ are obtained by covariantization of the flat derivative, $\partial^a \rightarrow \mathcal{D}^a$, and the linearized Einstein tensor, $G_{\text{lin}}^{ab} \rightarrow \hat{G}^{ab}$, in the respective contributions $\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_5$ to flat Lagrangian (2.31). Note also that contribution \mathcal{L}_6 (2.37) in the flat Lagrangian is promoted to the interacting Lagrangian without any changes (see (3.10)).

ii) Comparing \mathcal{L}_2 (3.6) that enters interacting Lagrangian (3.4) and the respective \mathcal{L}_2 (2.33) that enters flat Lagrangian (2.31), we see that \mathcal{L}_2 in (3.6) involves the additional contributions of first order in the shifted curvatures and second order in the field ϕ_0^{ab} . We note the $\mathcal{L}_2|_{b^a=0}$ part of \mathcal{L}_2 (3.6) is obtained simply by expanding the Einstein-Hilbert Lagrangian $\sqrt{g}R(g)$ with $g_{\mu\nu} = g_{\mu\nu} + \phi_{0,\mu\nu}$ as a power series in the field $\phi_{0,\mu\nu}$ and taking terms of the second order in the field $\phi_{0,\mu\nu}$, where $\phi_{0,\mu\nu} = e_{\mu a} e_{\nu b} \phi_0^{ab}$. So, we see that the tensor field ϕ_0^{ab} looks like an excitation of the graviton field.

iii) Interesting contribution which is absent in flat Lagrangian (2.31) and involved in interacting Lagrangian (3.4) is governed by the contribution denoted by \mathcal{L}_7 (3.11). From (3.11), we see that the tensor field ϕ_0^{ab} is coupled to energy-momentum tensor of the compensator field b^a , i.e., we see that the tensor field ϕ_0^{ab} again exhibits some properties of an excitation of the graviton field.

iv) The remaining contribution which is absent in the flat Lagrangian and enters the interacting Lagrangian is governed by the contribution denoted by \mathcal{L}_8 (3.12). From (3.12), we see that the tensor field ϕ_0^{ab} and the scalar field ϕ_0 lead to the appearance of a cubic potential. As is well known, massless fields in flat space, with exception of a scalar field, do not allow cubic vertices without derivatives (see e.g. Ref.[22]). Also, we know that cubic vertices having derivative independent contributions are allowed for arbitrary spin massive fields in flat space (see e.g. [23]). Appearance of the derivative independent cubic potential for the tensor field ϕ_0^{ab} implies that the tensor field ϕ_0^{ab} exhibits some features of the massive field. Taking above-given discussion into account, we see that, on the one hand, the tensor field ϕ_0^{ab} exhibits some features of excitation of the graviton field, i.e. massless spin-2 field, and, on the other hand, the field ϕ_0^{ab} exhibits some features of massive field.

Gauge transformations. We now discuss gauge symmetries of Lagrangian (3.4). Because we promote all gauge symmetries of free fields of 6d conformal gravity to the interacting fields we introduce the same amount of gauge transformation parameters as in the free theory (see (2.50)),

$$\begin{array}{ccc} \xi_{-3}^a & \xi_{-1}^a & \xi_1^a \\ & \xi_{-2} & \xi_0 \end{array} \quad (3.29)$$

We note that Weyl dimensions of gauge transformation parameters (3.29) are given by

$$\begin{aligned} \Delta_{\xi_{k'}^a}^w &= 2 + k', & k' &= -3, -1, 1, \\ \Delta_{\xi_{k'}}^w &= 2 + k', & k' &= -2, 0. \end{aligned} \quad (3.30)$$

Note also that because we are using frame-like description we assume, as usually, the standard Lorentz gauge transformations of our fields in (3.1),

$$\begin{aligned} \delta_\lambda e_\mu^a &= \lambda^{ab} e_\mu^b, & \delta_\lambda \phi_{k'}^{ab} &= \lambda^{ac} \phi_{k'}^{cb} + \lambda^{bc} \phi_{k'}^{ac}, & k' &= 0, 2, \\ \delta_\lambda b^a &= \lambda^{ac} b^c, & \delta_\lambda \phi_1^a &= \lambda^{ac} \phi_1^c, & \delta_\lambda \phi_0 &= 0, \end{aligned} \quad (3.31)$$

$\lambda^{ab} = -\lambda^{ba}$. We now discuss gauge transformations associated with gauge transformations parameters (3.29) in turn.

ξ_{-3}^a **transformation (diffeomorphism transformations).** In our approach, the gauge transformation parameter which is responsible for diffeomorphism transformations is denoted by ξ_{-3}^a . The diffeomorphism transformations take the standard form

$$\delta_{\xi_{-3}} e_\mu^a = \xi_{-3} \partial e_\mu^a + e_\mu^a \partial_\mu \xi_{-3}^\nu, \quad (3.32)$$

$$\delta_{\xi_{-3}} \phi_{k'}^{ab} = \xi_{-3} \partial \phi_{k'}^{ab}, \quad k' = 0, 2, \quad (3.33)$$

$$\delta_{\xi_{-3}} b^a = \xi_{-3} \partial b^a, \quad (3.34)$$

$$\delta_{\xi_{-3}} \phi_1^a = \xi_{-3} \partial \phi_1^a, \quad (3.35)$$

$$\delta_{\xi_{-3}} \phi_0 = \xi_{-3} \partial \phi_0, \quad (3.36)$$

$$\xi_{-3} \partial \equiv \xi_{-3}^\mu \partial_\mu, \quad \xi_{-3}^\mu \equiv e^{\mu a} \xi_{-3}^a. \quad (3.37)$$

ξ_{-1}^a **gauge transformations.** In our approach, gauge transformations associated with the parameter ξ_{-1}^a are simultaneously realized as the gradient gauge transformation of the spin-2 field ϕ_0^{ab} and the Stueckelberg gauge transformation of the compensator b^a . The ξ_{-1}^a gauge transformations receive interaction dependent corrections in the interacting theory. This is to say that the ξ_{-1}^a gauge transformations take the form

$$\delta_{\xi_{-1}} e_\mu^a = 0, \quad (3.38)$$

$$\delta_{\xi_{-1}} \phi_0^{ab} = \mathcal{D}^a \xi_{-1}^b + \mathcal{D}^b \xi_{-1}^a, \quad (3.39)$$

$$\delta_{\xi_{-1}} \phi_2^{ab} = \mathcal{L}_{\xi_{-1}} \phi_0^{ab} + \phi_1^a \xi_{-1}^b + \phi_1^b \xi_{-1}^a - \frac{1}{2} \eta^{ab} \phi_1^c \xi_{-1}^c, \quad (3.40)$$

$$\delta_{\xi_{-1}} b^a = -\xi_{-1}^a, \quad (3.41)$$

$$\delta_{\xi_{-1}} \phi_1^a = -\frac{1}{2} \phi_0^{ab} \xi_{-1}^b - \frac{1}{2} F^{ab} \xi_{-1}^b + \frac{u}{2} \phi_0 \xi_{-1}^a, \quad (3.42)$$

$$\delta_{\xi_{-1}} \phi_0 = 0, \quad (3.43)$$

where the action of Lie derivative $\mathcal{L}_{\xi_{-1}}$ on the field ϕ_0^{ab} is defined to be

$$\mathcal{L}_{\xi_{-1}} \phi_0^{ab} \equiv \xi_{-1}^c \mathcal{D}^c \phi_0^{ab} + \mathcal{D}^a \xi_{-1}^c \phi_0^{cb} + \mathcal{D}^b \xi_{-1}^c \phi_0^{ca}. \quad (3.44)$$

Comparing these gauge transformations with free theory gauge transformations given in (2.53), (2.54), (2.56), we see that there are two types of interaction dependent contributions. The first ones given in (3.39) are obtained simply by the covariantization of the flat derivatives in free theory transformations, $\partial^a \rightarrow \mathcal{D}^a$, (see (2.53)). The remaining contributions given in (3.40), (3.42) are obtained in due course of building both the interacting gauge invariant Lagrangian and the corresponding gauge transformations.

ξ_1^a gauge transformations. In our approach, gauge transformations associated with the parameter ξ_1^a are simultaneously realized as the gradient gauge transformation of the spin-2 field ϕ_2^{ab} and the Stueckelberg gauge transformation of the vector field ϕ_1^a . In the interacting theory, the ξ_1^a gauge transformations take the form

$$\delta_{\xi_1} e_\mu^a = 0, \quad (3.45)$$

$$\delta_{\xi_1} \phi_0^{ab} = 0, \quad (3.46)$$

$$\delta_{\xi_1} \phi_2^{ab} = \mathcal{D}^a \xi_1^b + \mathcal{D}^b \xi_1^a, \quad (3.47)$$

$$\delta_{\xi_1} b^a = 0, \quad (3.48)$$

$$\delta_{\xi_1} \phi_1^a = -\xi_1^a, \quad (3.49)$$

$$\delta_{\xi_1} \phi_0 = 0. \quad (3.50)$$

Comparing these gauge transformations with free theory gauge transformations given in (2.54), (2.56), we see that all that is required for the generalization of free theory ξ_1^a gauge transformations is to make covariantization of the gradient gauge transformation of the spin-2 field ϕ_2^{ab} (see (2.54) and (3.47))

ξ_{-2} gauge transformations (Weyl gauge transformations). In our approach, gauge transformation parameter responsible for Weyl gauge transformations is denoted by ξ_{-2} . To make contact with the commonly used notation we introduce the parameter σ by the relation

$$\sigma = -\frac{1}{4} \xi_{-2}. \quad (3.51)$$

Using Weyl dimensions of our fields (3.3), we write down the standard Weyl gauge transformations for our fields

$$\delta_{\xi_{-2}} e_\mu^a = -\sigma e_\mu^a, \quad (3.52)$$

$$\delta_{\xi_{-2}} \phi_{k'}^{ab} = (2 + k') \sigma \phi_{k'}^{ab}, \quad k' = 0, 2, \quad (3.53)$$

$$\delta_{\xi_{-2}} b^a = D^a \xi_{-2} + \sigma b^a, \quad (3.54)$$

$$\delta_{\xi_{-2}} \phi_1^a = 3\sigma \phi_1^a. \quad (3.55)$$

$$\delta_{\xi_{-2}} \phi_0 = 2\sigma \phi_0. \quad (3.56)$$

ξ_0 gauge transformations. In our approach, gauge transformations associated with parameter ξ_0 are simultaneously realized as the gradient gauge transformation for the spin-1 field ϕ_1^a and the Stueckelberg transformation of the scalar field ϕ_0 . In the interacting theory, the ξ_0 gauge transformations take the form

$$\delta_{\xi_0} e_\mu^a = 0, \quad (3.57)$$

$$\delta_{\xi_0} \phi_0^{ab} = \frac{1}{2} \eta^{ab} \xi_0, \quad (3.58)$$

$$\delta_{\xi_0} \phi_2^{ab} = -\frac{1}{2} \phi_0^{ab} \xi_0, \quad (3.59)$$

$$\delta_{\xi_0} b^a = 0, \quad (3.60)$$

$$\delta_{\xi_0} \phi_1^a = \mathcal{D}^a \xi_0, \quad (3.61)$$

$$\delta_{\xi_0} \phi_0 = -u \xi_0. \quad (3.62)$$

Comparing these gauge transformations with free theory gauge transformations given in (2.54), (2.56), we see that the generalization of free theory ξ_0 gauge transformations is arrived in two steps: i) by covariantization of the gradient gauge transformations of spin-1 field ϕ_1^a (see (2.56) and (3.61)); ii) by modification of the gauge transformation of field ϕ_2^{ab} (see (2.54) and (3.59)).

Matching of linearized background gauge symmetries of interacting theory and global symmetries of free theory. Let us recall the definition of linearized background gauge symmetries. Consider a gauge transformation for interacting field Φ ,

$$\delta \Phi = G_\alpha(\Phi) \xi^\alpha. \quad (3.63)$$

If $\bar{\Phi}$ is a solution to equations of motion, then gauge transformation that respects this solution is realized by using the gauge transformation parameters $\bar{\xi}^\alpha$ satisfying the equations

$$G_\alpha(\bar{\Phi}) \bar{\xi}^\alpha = 0. \quad (3.64)$$

Using the field expansion $\Phi = \bar{\Phi} + \phi$, the linearized background gauge transformations are then defined as

$$\delta \phi = \partial_\Phi G_\alpha(\bar{\Phi}) \bar{\xi}^\alpha \phi, \quad (3.65)$$

where ∂_Φ stands for a functional derivative. As is well known, the linearized background gauge transformations are interrelated with global symmetries of the corresponding flat theory. We now demonstrate this interrelation for the case of $6d$ conformal gravity.

We note that solution to $6d$ conformal gravity equations of motions corresponding to the flat space background is given by

$$\bar{e}_\mu^a = \delta_\mu^a, \quad \bar{\phi}_0^{ab} = 0, \quad \bar{\phi}_2^{ab} = 0, \quad \bar{b}^a = 0, \quad \bar{\phi}_0 = 0. \quad (3.66)$$

Collecting all gauge transformations

$$\delta = \delta_{\xi_{-3}^a} + \delta_{\xi_{-1}^a} + \delta_{\xi_1^a} + \delta_{\xi_{-2}} + \delta_{\xi_0} + \delta_{\lambda^{ab}} \quad (3.67)$$

and using notation $\bar{\Phi}$ for background fields in (3.66), we now look for gauge transformations that respect solution given in (3.66)

$$\delta \bar{\Phi} = 0, \quad (3.68)$$

To discuss solution to equations in (3.68) we use the following notation. Solution to gauge transformation parameter ξ that corresponds to symmetry generator G will be denoted as ξ^G . In our case there are the following set of symmetry generators $G = P^a, J^{ab}, D, K^a$. We now write solutions to gauge transformation parameters corresponding to these symmetry generators.

Poincaré translations,

$$\begin{aligned} \bar{\xi}_{-3}^{bP^a} &= \eta^{ab}, & \bar{\xi}_{-1}^{bP^a} &= 0, & \bar{\xi}_1^{bP^a} &= 0, \\ \bar{\xi}_{-2}^{P^a} &= 0, & \bar{\xi}_0^{P^a} &= 0, & \bar{\lambda}^{bcP^a} &= 0; \end{aligned} \quad (3.69)$$

Lorentz rotations,

$$\begin{aligned} \bar{\xi}_{-3}^{cJ^{ab}} &= 2\eta^{c[b}x^{a]}, & \bar{\xi}_{-1}^{cJ^{ab}} &= 0, & \bar{\xi}_1^{cJ^{ab}} &= 0, \\ \bar{\xi}_{-2}^{J^{ab}} &= 0, & \bar{\xi}_0^{J^{ab}} &= 0, & \bar{\lambda}^{ce,J^{ab}} &= 2\eta^{a[c}\eta^{e]b}; \end{aligned} \quad (3.70)$$

Dilatation,

$$\begin{aligned} \bar{\xi}_{-3}^{aD} &= x^a, & \bar{\xi}_{-1}^{aD} &= 0, & \bar{\xi}_1^{aD} &= 0, \\ \bar{\xi}_{-2}^D &= -4, & \bar{\xi}_0^D &= 0, & \bar{\lambda}^{abD} &= 0; \end{aligned} \quad (3.71)$$

Conformal boosts,

$$\begin{aligned} \bar{\xi}_{-3}^{bK^a} &= -\frac{1}{2}x^2\eta^{ab} + x^ax^b, & \bar{\xi}_{-1}^{bK^a} &= -4\eta^{ab}, & \bar{\xi}_1^{bK^a} &= 0, \\ \bar{\xi}_{-2}^{K^a} &= -4x^a, & \bar{\xi}_0^{K^a} &= 0, & \bar{\lambda}^{bc,K^a} &= 2\eta^{a[b}x^{c]}. \end{aligned} \quad (3.72)$$

We now note that the linearized background gauge symmetries with gauge transformation parameters given in (3.69), (3.70), (3.71), and (3.72) correspond to the respective Poincaré translation, Lorentz, dilatation, and conformal boost symmetries of $6d$ conformal gravity in the flat space. To demonstrate this, we introduce fields of $6d$ conformal gravity in flat space background (3.68),

$$\phi_{-2}^{ab}, \quad \phi_0^{ab}, \quad \phi_2^{ab}, \quad \phi_{-1}^a, \quad \phi_1^a, \quad \phi_0, \quad (3.73)$$

where ϕ_{-2}^{ab} appears in the small field expansion of the vielbein field e_μ^a , while ϕ_{-1}^a is identified with the compensator field in flat space background,

$$e_\mu^a = \delta_\mu^a + \frac{1}{2}\phi_{-2}^{ab}\eta_{\mu b}, \quad b^a = \phi_{-1}^a. \quad (3.74)$$

Note that expansion for e_μ^a (3.74) implies that we use Lorentz gauge transformation (3.31) to get symmetric tensor field ϕ_{-2}^{ab} . Now using gauge transformation parameters given in (3.69), (3.70),

(3.71) and general relation given in (3.65) we make sure that linearized background gauge symmetries with gauge transformation parameters in (3.69), (3.70) and (3.71) coincide precisely with the respective Poincaré and dilatation symmetries of free $6d$ conformal theory discussed in Sec. 2.2. Matching of the conformal boost symmetries turn out to be more interesting. This is to say that the linearized background gauge symmetries with gauge transformation parameters given in (3.72) take the form given in (2.66) with the same $K_{\Delta,M}^a$ transformations as in (2.67) and the following R^a transformations:

$$\delta_{R^a} \phi_{-2}^{bc} = 0, \quad (3.75)$$

$$\begin{aligned} \delta_{R^a} \phi_0^{bc} &= -2\eta^{ab} \phi_{-1}^c - 2\eta^{ac} \phi_{-1}^b + 2\eta^{bc} \phi_{-1}^a \\ &\quad - 4\partial^a \phi_{-2}^{bc} + 2\partial^b \phi_{-2}^{ac} + 2\partial^c \phi_{-2}^{ab}, \end{aligned} \quad (3.76)$$

$$\delta_{R^a} \phi_2^{bc} = -4\eta^{ab} \phi_1^c - 4\eta^{ac} \phi_1^b + 2\eta^{bc} \phi_1^a - 4\partial^a \phi_0^{bc}, \quad (3.77)$$

$$\delta_{R^a} \phi_{-1}^b = 2\phi_{-2}^{ab}, \quad (3.78)$$

$$\delta_{R^a} \phi_1^b = 2\phi_0^{ab} - 2\eta^{ab} u \phi_0 - 2F^{ab}(\phi_{-1}), \quad (3.79)$$

$$\delta_{R^a} \phi_0 = 0. \quad (3.80)$$

Comparing R^a transformations in (2.70)-(2.75) and the ones in (3.75)-(3.80), we notice some differences in R^a transformations for the fields ϕ_0^{bc} , ϕ_{-1}^b , ϕ_1^b , and ϕ_0 . Explanation of these differences is obvious: global transformations of gauge fields are defined up to gauge transformations. Introducing notation for the gauge transformation parameters

$$\xi_{-1}^{aK^b} \equiv 2\phi_{-2}^{ab}, \quad \xi_0^{K^a} = 2\phi_{-1}^a, \quad (3.81)$$

and using notation $\delta_{R^a}^{\text{flat}}$ and δ_{R^a} for the respective R^a transformation in (2.70)-(2.75) and (3.75)-(3.80), we note that R^a transformations given in (2.70)-(2.75) and the ones in (3.75)-(3.80) are related by the gauge transformations,

$$\delta_{R^a} \phi_0^{bc} = \delta_{R^a}^{\text{flat}} \phi_0^{bc} + \partial^b \xi_{-1}^{cK^a} + \partial^c \xi_{-1}^{bK^a} + \frac{1}{2} \eta^{bc} \xi_0^{K^a}, \quad (3.82)$$

$$\delta_{R^a} \phi_{-1}^b = \delta_{R^a}^{\text{flat}} \phi_{-1}^b - \xi_{-1}^{bK^a}, \quad (3.83)$$

$$\delta_{R^a} \phi_1^a = \delta_{R^a}^{\text{flat}} \phi_1^a + \partial^b \xi_0^{bK^a}, \quad (3.84)$$

$$\delta_{R^a} \phi_0 = \delta_{R^a}^{\text{flat}} \phi_0 - u \xi_0^{K^a}. \quad (3.85)$$

Thus, we see that the conformal boost transformations also match.

4 Higher-derivative Lagrangian of interacting $6d$ conformal gravity

Our ordinary-derivative Lagrangian can be used for the derivation of the higher-derivative Lagrangian of interacting $6d$ conformal gravity. The higher-derivative Lagrangian of interacting theory can be obtained by following the procedure we used for the derivation of the higher-derivative

Lagrangian of free 6d conformal gravity in Sec.(2.2). We now proceed to details of the derivation.

From gauge transformations (3.41),(3.49),(3.62), we see that the vector fields b^a , ϕ_1^a and the scalar field ϕ_0 transform as Stueckelberg fields and can therefore be gauged away by fixing the Stueckelberg gauge symmetries. Gauging away the vector fields and the scalar field,

$$b^a = 0, \quad \phi_1^a = 0, \quad \phi_0 = 0, \quad (4.1)$$

we see that our Lagrangian (3.4) takes the simplified form

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_6 + \mathcal{L}_8, \quad (4.2)$$

$$e^{-1}\mathcal{L}_1 = -\phi_2^{ab}G^{ab}, \quad (4.3)$$

$$\begin{aligned} e^{-1}\mathcal{L}_2 = & -\frac{1}{4}D^a\phi_0^{bc}D^a\phi_0^{bc} + \frac{1}{8}D^a\phi_0^{bb}D^a\phi_0^{cc} + \frac{1}{2}C_1^aC_1^a \\ & - \frac{1}{2}R^{cabe}\phi_0^{ab}\phi_0^{ce} + \frac{1}{2}R^{ab}\phi_0^{ac}\phi_0^{cb} - \frac{1}{2}R^{ab}\phi_0^{ab}\phi_0^{cc} + \left(\frac{1}{8}\phi_0^{aa}\phi_0^{bb} - \frac{1}{4}\phi_0^{ab}\phi_0^{ab}\right)R, \end{aligned} \quad (4.4)$$

$$e^{-1}\mathcal{L}_6 = -\frac{1}{2}\phi_2^{ab}\chi_0^{ab}, \quad (4.5)$$

$$e^{-1}\mathcal{L}_8 = \frac{1}{4}\phi_0^{ab}\phi_0^{bc}\phi_0^{ca} - \frac{5}{16}\phi_0^{ab}\phi_0^{ab}\phi_0^{cc} + \frac{1}{16}(\phi_0^{aa})^3, \quad (4.6)$$

$$G^{ab} \equiv R^{ab} - \frac{1}{2}\eta^{ab}R, \quad (4.7)$$

$$C_1^a \equiv D^b\phi_0^{ab} - \frac{1}{2}D^a\phi_0^{cc}, \quad (4.8)$$

$$\chi_0^{ab} \equiv \phi_0^{ab} - \eta^{ab}\phi_0^{cc}. \quad (4.9)$$

We note that the Lagrangian (4.2) depends on the rank-2 tensor field ϕ_2^{ab} linearly (see expressions for \mathcal{L}_1 and \mathcal{L}_6). Using equations of motion for the field ϕ_2^{ab} obtained from Lagrangian (4.2) we find the equation

$$\phi_0^{ab} - \eta^{ab}\phi_0^{cc} = -2G^{ab}, \quad (4.10)$$

which has obvious solution

$$\bar{\phi}_0^{ab} = -2R^{ab} + \frac{1}{5}\eta^{ab}R. \quad (4.11)$$

Plugging solution $\bar{\phi}_0^{ab}$ (4.11) into Lagrangian (4.2) we obtain the higher-derivative Lagrangian

$$e^{-1}\mathcal{L} = R^{ab}D^2R^{ab} - \frac{3}{10}RD^2R - 2R^{cabe}R^{ab}R^{ce} - R^{ab}R^{ab}R + \frac{3}{25}R^3. \quad (4.12)$$

This higher-derivative Lagrangian should be invariant under Weyl gauge transformations

$$\delta e_\mu^a = -\sigma e_\mu^a. \quad (4.13)$$

We have checked directly that the Lagrangian is indeed invariant under Weyl gauge transformations (4.13). Note that by using various identities for R^3 terms (see e.g. Ref.[24]), the Lagrangian can be expressed in terms of the Weyl tensor and Ricci curvatures.

Though there is a lot of literature on conformal gravity, we did not find discussion of the Lagrangian (4.12) in the earlier literature. We note however that all Weyl invariant densities for $6d$ conformal theory were presented in Ref.[25] (see also Refs.[26, 27, 28]¹¹). Up to total derivative, in $6d$ conformal gravity theory, there are three Weyl invariant densities constructed out the Weyl tensor, Ricci curvatures, and covariant derivative. In Ref.[25], the simplest combination of those three invariants, which involves $R^{ab}D^2R^{ab}$ term, has been found (see Eq.(4.12) in Ref.[25]). It turns out that it is this simplest invariant that coincides with our Lagrangian in (4.12)¹².

As a side of remark we note that the remaining two Weyl invariant densities also can be lifted to our gauge-invariant approach, i.e., it is possible to build the invariants which respect all gauge symmetries of our approach. To this end we introduce the new curvature

$$\mathcal{R}^{abce} = R^{abce} + \eta^{ac}\psi^{be} - \eta^{bc}\psi^{ae} + \eta^{be}\psi^{ac} - \eta^{ae}\psi^{bc}, \quad (4.14)$$

$$\psi^{ab} \equiv \frac{q}{2} \left(\phi_0^{ab} + D^a b^b + D^b b^a + 2qb^a b^b - q\eta^{ab} b^2 \right), \quad (4.15)$$

$q = 1/4$, and note that under ξ_{-2} , ξ_{-1}^a , and ξ_1^a gauge transformations new curvature \mathcal{R}^{abce} (4.14) transforms as

$$\delta_{\xi_{-2}} \mathcal{R}^{abce} = 2\sigma \mathcal{R}^{abce}, \quad \delta_{\xi_{-1}} \mathcal{R}^{abce} = 0, \quad \delta_{\xi_1} \mathcal{R}^{abce} = 0, \quad (4.16)$$

$$\delta_{\xi_0} \mathcal{R}^{abce} = \frac{q}{2} (\eta^{ac}\eta^{be} - \eta^{ae}\eta^{bc}) \xi_0. \quad (4.17)$$

From these relations, we see that the curvature \mathcal{R}^{abce} has Weyl dimension equal to 2 and this curvature is invariant under the ξ_{-1}^a and ξ_1^a gauge transformations. Also we note that gauging away Stueckelberg fields (4.1) and using solution (4.11), the curvature \mathcal{R}^{abce} becomes the standard Weyl tensor

$$\mathcal{R}^{abce}|_{b^a=0, \phi_0^{ab}=\bar{\phi}_0^{ab}} = C^{abce}. \quad (4.18)$$

We see however that the curvature \mathcal{R}^{abce} does not respect ξ_0 gauge symmetry (4.17). General curvature that respects the ξ_0 gauge symmetry can be built as follows

$$\begin{aligned} \mathbf{R}^{abce} &= \mathcal{R}^{abce} + h_1(\eta^{ac}\mathcal{R}^{be} - \eta^{bc}\mathcal{R}^{ae} + \eta^{be}\mathcal{R}^{ac} - \eta^{ae}\mathcal{R}^{bc}) \\ &+ h_2(\eta^{ac}\eta^{be} - \eta^{ae}\eta^{bc})\mathcal{R} + h_3(\eta^{ac}\eta^{be} - \eta^{ae}\eta^{bc})\phi_0, \end{aligned} \quad (4.19)$$

where $\mathcal{R}^{ab} = \mathcal{R}^{cab}$, $\mathcal{R} = \mathcal{R}^{aa}$ and coefficients h_1, h_2, h_3 satisfy the equation

$$1 + 10h_1 + 30h_2 - \frac{2u}{q}h_3 = 0, \quad (4.20)$$

¹¹ Discussion of interesting methods for constructing Weyl invariant densities may be found in [29, 30]. Classification of all the six-derivative Lagrangians in arbitrary dimensions such that the trace of the resulting field equations are at most of order 3 may be found in Ref.[31]. Study of the effective $6d$ conformal gravity may be found in Ref.[32].

¹² In (4.12), our signs in front of terms involving odd number of the Ricci tensors and Ricci scalars are opposite to the ones in Ref.[25]. Perhaps these sign differences can be explained by the different conventions used for the definition of the Ricci tensor and Ricci scalar in our paper and in Ref.[25]. Our curvature conventions are $R^\lambda{}_{\mu\nu\rho} = \partial_\nu \Gamma^\lambda_{\mu\rho} - \dots$, $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$, $R = R^\mu{}_\mu$.

and u is given in (2.21). Equation (4.20) is simply obtained by requiring the curvature \mathbf{R}^{abce} to be invariant under ξ_0 gauge transformations (3.57)-(3.62). Thus curvature (4.19) has the desired properties

$$\delta_{\xi_{-2}} \mathbf{R}^{abce} = 2\sigma \mathbf{R}^{abce}, \quad \delta_{\xi_{-1}} \mathbf{R}^{abce} = 0, \quad \delta_{\xi_1} \mathbf{R}^{abce} = 0, \quad \delta_{\xi_0} \mathbf{R}^{abce} = 0, \quad (4.21)$$

Using the curvature \mathbf{R}^{abce} , we can construct the remaining two gauge invariant densities in a straightforward way. For instance, we can consider the invariants

$$e \mathbf{R}_{ce}^{ab} \mathbf{R}_{fg}^{ce} \mathbf{R}_{ab}^{fg}, \quad e \mathbf{R}_{ef}^{ab} \mathbf{R}_{fg}^{bc} \mathbf{R}_{ge}^{ca}. \quad (4.22)$$

In our gauge invariant approach, these invariants are counterparts of the well known Weyl invariants appearing in the standard approach to $6d$ conformal theory,

$$e C_{ce}^{ab} C_{fg}^{ce} C_{ab}^{fg}, \quad e C_{ef}^{ab} C_{fg}^{bc} C_{ge}^{ca}, \quad (4.23)$$

i.e. by using Stueckelberg gauge conditions (4.1) and pugging solution (4.11) into invariants (4.22), we obtain the respective Weyl invariant densities given in (4.23).

Thus, the remaining two invariants (4.22) respect all gauge symmetries of our gauge invariant approach. The important difference of these two invariants as compared to our Lagrangian (3.4) is that these two invariants involve the higher-derivatives, while our Lagrangian involves only the ordinary derivatives.¹³ At present time, we do not know representation of the invariants (4.22) in terms of the ordinary derivatives. It is not obvious that such representation can be constructed without adding new fields to our field content in (3.1).

We finish with remark that some special representatives of general curvature (4.19) might be interesting in various contexts. For instance, solution to equation (4.20) given by

$$h_1 = -\frac{1}{4}, \quad h_2 = \frac{1}{20}, \quad h_3 = 0, \quad (4.24)$$

leads to traceless tensor

$$\begin{aligned} \mathbf{R}_{\text{weyl}}^{abce} &= \mathcal{R}^{abce} - \frac{1}{4}(\eta^{ac}\mathcal{R}^{be} - \eta^{bc}\mathcal{R}^{ae} + \eta^{be}\mathcal{R}^{ac} - \eta^{ae}\mathcal{R}^{bc}) \\ &+ \frac{1}{20}(\eta^{ac}\eta^{be} - \eta^{ae}\eta^{bc})\mathcal{R}, \end{aligned} \quad (4.25)$$

which can be considered as counterpart of the Weyl tensor in our approach. Another solution to equation (4.20) given by

$$h_1 = 0, \quad h_2 = 0, \quad h_3 = \frac{q}{2u}, \quad (4.26)$$

leads to the curvature

$$\mathbf{R}_{\text{scal}}^{abce} = \mathcal{R}^{abce} + \frac{q}{2u}(\eta^{ac}\eta^{be} - \eta^{ae}\eta^{bc})\phi_0. \quad (4.27)$$

The curvature $\mathbf{R}_{\text{scal}}^{abce}$ has the following interesting property. If we introduce the corresponding Einstein tensor

$$\mathbf{G}_{\text{scal}}^{ab} = \mathbf{R}_{\text{scal}}^{ab} - \frac{1}{2}\eta^{ab}\mathbf{R}_{\text{scal}}, \quad (4.28)$$

¹³ In Ref.[33] (see Sec.8.2), authors conjectured that two invariants given in (4.23) do not deform gauge algebra in the standard higher-derivative approach to conformal $6d$ gravity. The fact that our remaining two invariants (4.22) respect our Stueckelberg gauge symmetries seems to be in agreement with this conjecture.

where $\mathbf{R}_{\text{scal}}^{ab} = \mathbf{R}_{\text{scal}}^{cab}$, $\mathbf{R}_{\text{scal}} = \mathbf{R}_{\text{scal}}^{aa}$, then it turns out that the \mathcal{L}_1 and \mathcal{L}_6 parts of the Lagrangian (3.4) can be collected as

$$\mathcal{L}_1 + \mathcal{L}_6 = -\phi_2^{ab} \mathbf{G}_{\text{scal}}^{(ab)}. \quad (4.29)$$

Because the field ϕ_2^{ab} does not appear in the remaining contributions to Lagrangian (3.4), equations of motion for this field can be represented as

$$\mathbf{G}_{\text{scal}}^{(ab)} = 0. \quad (4.30)$$

5 Conclusions

In this paper, we applied the ordinary-derivative approach, developed in Ref.[6], to the study of $6d$ interacting conformal gravity. The results presented here should have a number of interesting applications and generalizations. Let us comment on some of them.

i) As we have already mentioned, the gauge symmetries of our Lagrangian make it possible to match our approach with the standard one, i.e., by an appropriate gauge fixing of the Stueckelberg fields and solving the constraints, we obtain the higher-derivative formulation of the $6d$ conformal gravity. This implies that, at least at the classical level, our $6d$ conformal gravity theory is equivalent to the standard one. In this respect it would be interesting to investigate quantum equivalence of our theory and the standard one. We note that our formulation provides new interesting possibilities for investigation of quantum behavior of conformal gravity. The first step in studying quantum behavior of conformal gravity is a computation of one-loop effective action. One powerful method of the computation of the one-loop effective action of Einstein gravity is based on the use of so called Δ_2 algorithm [34, 35, 36] (see also [37]¹⁴). In order to investigate quantum properties of fourth-derivative $4d$ Weyl gravity this algorithm was generalized to the so called Δ_4 -algorithm (see [39] and refs. there). We note, however, that since our approach does not involve higher-derivatives and formulated in terms of conventional kinetic terms we can use the standard Δ_2 algorithm for the investigation of the one-loop effective action.

ii) In addition to the local Weyl and diffeomorphism symmetries that enter the standard approach to conformal gravity our approach involves gauge symmetries for two rank-2 tensor fields and some amount of Stueckelberg gauge symmetries. In other words, we deal with extended gauge algebra. In this respect, it would be interesting to analyze the general solution of the Wess-Zumino consistency condition for our gauge algebra along the lines in Ref.[30].

iii) Results in this paper provide the complete ordinary-derivative description of interacting $6d$ conformal gravity. It would be interesting to apply these results to the study of supersymmetric conformal field theories [40]-[44] in the framework of ordinary-derivative approach. The first step in this direction would be understanding of how the supersymmetries are realized in the framework of our approach.

iv) BRST approach is one of powerful approaches to the analysis of various aspects of relativistic dynamics (see e.g. Refs.[45]-[50]). This approach turned out to be successful for application to string theory. We believe therefore that use of this approach for the study of conformal fields might also be helpful for the better understanding of conformal gravity theory.

v) In the last years, there were interesting developments in the studying mixed-symmetry fields [51]-[59] that are invariant with respect to anti-de Sitter or Minkowski space-time symmetries. It would be interesting to apply methods developed in Refs.[51]-[59] to the studying interacting $6d$

¹⁴ Generalization of methods in Refs.[34, 35, 36] to quantum effective actions for brane induced gravity models may be found in Refs.[38].

conformal mixed-symmetry fields¹⁵. There are other various interesting approaches in the literature which could be used to discuss the ordinary-derivative formulation of $6d$ conformal fields. This is to say that various recently developed interesting formulations in terms of unconstrained fields in flat space may be found e.g. in Refs.[60]-[63].

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Appendix A Notation

Flat space-time notation. Our conventions are as follows. Coordinates in the flat space-time are denoted by x^a , while ∂_a stands for derivative with respect to x^a , $\partial_a \equiv \partial/\partial x^a$. Vector indices of the Lorentz algebra $so(5, 1)$ take the values $a, b, c, e = 0, 1, \dots, 5$. To simplify our expressions we drop the flat metric η_{ab} in scalar products, i.e. we use $X^a Y^a \equiv \eta_{ab} X^a Y^b$. We use operators constructed out of the coordinates and derivatives,

$$\square = \partial^a \partial^a, \quad x\partial = x^a \partial^a. \quad (\text{A.1})$$

Curved space-time notation. We use space-time base manifold indices $\mu, \nu, \rho, \sigma = 0, 1, \dots, 5$ and tangent-flat vectors indices of the $so(5, 1)$ algebra $a, b, c, e, f = 0, 1, \dots, 5$. Base manifold coordinates are denoted by x^μ , while ∂_μ denote the respective derivatives, $\partial_\mu \equiv \partial/\partial x^\mu$. We use notation e_μ^a and $\omega_\mu^{ab}(e)$ for the respective vielbein and the Lorentz connection. Contravariant tensor field carrying base manifold indices, $\phi^{\mu_1 \dots \mu_s}$, is related to tensor field carrying the tangent-flat indices, $\phi^{a_1 \dots a_s}$, in a standard way $\phi^{a_1 \dots a_s} \equiv e_{\mu_1}^{a_1} \dots e_{\mu_s}^{a_s} \phi^{\mu_1 \dots \mu_s}$. Covariant derivative D_μ , acting on vector field

$$D_\mu \phi^a = \partial_\mu \phi^a + \omega_\mu^{ab}(e) \phi^b, \quad (\text{A.2})$$

satisfies the standard commutator

$$[D_\mu, D_\nu] \phi^a = R_{\mu\nu}{}^{ab} \phi^b. \quad (\text{A.3})$$

Instead of D_μ , we prefer to use a covariant derivative with the flat indices D^a ,

$$D_a \equiv e_a^\mu D_\mu, \quad D^a = \eta^{ab} D_b, \quad (\text{A.4})$$

where e_a^μ is inverse of the vielbein, $e_\mu^a e_b^\mu = \delta_b^a$.

For field ϕ^a with Weyl dimension Δ_ϕ^w , covariant derivative defined in (3.20) satisfies the commutator

$$[\mathcal{D}^a, \mathcal{D}^b] \phi^c = \widehat{R}^{abce} \phi^e + q \Delta_\phi^w F^{ab}(b) \phi^c, \quad q = 1/4, \quad (\text{A.5})$$

where the shifted curvature and field strength F^{ab} are defined in (3.26) and (3.19) respectively. Weyl gauge transformations defined in (3.51)-(3.56) lead to the following transformation rules:

$$\delta \omega^{abc} = \sigma \omega^{abc} + \eta^{ac} D^b \sigma - \eta^{ab} D^c \sigma, \quad (\text{A.6})$$

$$\delta \widehat{R}^{abce} = 2\sigma \widehat{R}^{abce}, \quad (\text{A.7})$$

$$\delta \widehat{\omega}^{abc} = \sigma \widehat{\omega}^{abc}, \quad (\text{A.8})$$

¹⁵ Unfolded form of equations of motion for conformal mixed-symmetry fields is studied in Ref.[4]. Higher-derivative Lagrangian formulation of the mixed-symmetry conformal fields was recently developed in Ref.[2].

$$\delta \mathcal{F}^{ab}(\phi) = (\Delta_\phi^w + 1) \sigma \mathcal{F}^{ab}(\phi), \quad (\text{A.9})$$

$$\delta F^{ab} = 2\sigma F^{ab}, \quad (\text{A.10})$$

$$\delta(\mathcal{D}^a \phi^{a_1 \dots a_s}) = (\Delta_\phi^w + 1) \mathcal{D}^a \phi^{a_1 \dots a_s}. \quad (\text{A.11})$$

These relations imply the following Weyl dimensions:

$$\Delta_{\hat{R}^{abce}}^w = 2, \quad \Delta_{\hat{\omega}^{abc}}^w = 1, \quad \Delta_{F^{ab}}^w = 2, \quad \Delta_{\mathcal{F}^{ab}(\phi)}^w = \Delta_\phi^w + 1. \quad (\text{A.12})$$

Bianchi identities for the shifted curvature and field strength (3.19),

$$\mathcal{D}^f \hat{R}^{abce} + \text{cycl.perms.}(fab) = 0, \quad (\text{A.13})$$

$$\mathcal{D}^a F^{bc} + \text{cycl.perms.}(abc) = 0, \quad (\text{A.14})$$

can be obtained in a usual way. Using explicit relations for the curvatures and the Einstein tensor

$$\hat{R}^{abce} = R^{abce} + \eta^{ac} \varphi^{be} - \eta^{bc} \varphi^{ae} + \eta^{be} \varphi^{ac} - \eta^{ae} \varphi^{bc}, \quad (\text{A.15})$$

$$\hat{R}^{ab} = \hat{R}^{cab}, \quad \hat{R} = \hat{R}^{aa}, \quad (\text{A.16})$$

$$\hat{R}^{ab} = R^{ab} + 4\varphi^{ab} + \eta^{ab} \varphi^{cc}, \quad (\text{A.17})$$

$$\hat{R} = R + 10\varphi^{aa}, \quad (\text{A.18})$$

$$\hat{G}^{ab} \equiv \hat{R}^{ab} - \frac{1}{2} \eta^{ab} \hat{R}, \quad (\text{A.19})$$

$$\varphi^{ab} = q D^a b^b + q^2 b^a b^b - \frac{1}{2} q^2 \eta^{ab} b^2, \quad (\text{A.20})$$

$$\varphi^{cc} = q D b - 2q^2 b^2, \quad q = 1/4, \quad (\text{A.21})$$

we obtain various useful identities

$$\hat{R}^{abce} - \hat{R}^{ceab} = q(\eta^{ac} F^{be} - \eta^{bc} F^{ae} + \eta^{be} F^{ac} - \eta^{ae} F^{bc}), \quad (\text{A.22})$$

$$\hat{R}^{ab} - \hat{R}^{ba} = F^{ab}, \quad (\text{A.23})$$

$$\mathcal{D}^c \hat{R}^{abce} = \mathcal{D}^a \hat{R}^{be} - \mathcal{D}^b \hat{R}^{ae}, \quad (\text{A.24})$$

$$\begin{aligned} \mathcal{D}^c \hat{R}^{ceab} &= \mathcal{D}^a \hat{R}^{be} - \mathcal{D}^b \hat{R}^{ae} \\ &\quad + q(\mathcal{D}^e F^{ab} - \eta^{ae} \mathcal{D}^c F^{cb} + \eta^{be} \mathcal{D}^c F^{ca}), \end{aligned} \quad (\text{A.25})$$

$$\mathcal{D}^b \hat{R}^{ab} = \frac{1}{2} \mathcal{D}^a \hat{R}, \quad (\text{A.26})$$

$$\mathcal{D}^b \hat{R}^{ba} = \frac{1}{2} \mathcal{D}^a \hat{R} + \mathcal{D}^b F^{ba}, \quad (\text{A.27})$$

$$\mathcal{D}^b \hat{G}^{ab} = 0, \quad \mathcal{D}^b \hat{G}^{ba} = \mathcal{D}^b F^{ba}, \quad \mathcal{D}^b \hat{G}^{(ab)} = \frac{1}{2} \mathcal{D}^b F^{ba}. \quad (\text{A.28})$$

Throughout this paper, symmetrization and antisymmetrization of the indices are normalized as $(ab) = \frac{1}{2}(ab + ba)$, $[ab] = \frac{1}{2}(ab - ba)$.

Appendix B Derivation of interacting gauge invariant Lagrangian

In this Appendix we outline some details of the derivation of gauge invariant Lagrangian given in (3.4) and the corresponding gauge transformations. We divide our derivation in seven steps which we now discuss in turn.

Step 1. We begin with the discussion of contribution to Lagrangian (3.4) denoted by \mathcal{L}_1 (3.5). This contribution is simply obtained from the contribution to free theory theory Lagrangian (2.31) also denoted by \mathcal{L}_1 (2.32) in a straightforward way. Namely \mathcal{L}_1 (3.5) is obtained from \mathcal{L}_1 (2.32) by requiring \mathcal{L}_1 (3.5) to be invariant under Weyl gauge transformations given in (3.52)-(3.56). All that is required to respect those gauge transformations is to replace the linearized shifted Einstein tensor and curvatures given in (2.43)-(2.49) by the corresponding complete shifted Einstein tensor and curvatures given in (3.28) and (A.17),(A.18).

Step 2. We covariantize the flat ξ_{-1}^a gauge transformation of the field ϕ_0^{ab} (2.53) by making replacement $\partial^a \rightarrow \mathcal{D}^a$, while the ξ_{-1}^a gauge transformation of the field b^a (2.55) is not changed,

$$\delta_{\xi_{-1}} \phi_0^{ab} = \mathcal{D}^a \xi_{-1}^b + \mathcal{D}^b \xi_{-1}^a, \quad (\text{B.1})$$

$$\delta_{\xi_{-1}} b^a = -\xi_{-1}^a. \quad (\text{B.2})$$

Also, making the covariantization $\partial^a \rightarrow \mathcal{D}^a$ in contribution to flat Lagrangian denoted by \mathcal{L}_2 (2.33), we introduce

$$e^{-1} \mathcal{L}_{2K} = -\frac{1}{4} \mathcal{D}^a \phi_0^{bc} \mathcal{D}^a \phi_0^{bc} + \frac{1}{8} \mathcal{D}^a \phi_0^{bb} \mathcal{D}^a \phi_0^{cc} + \frac{1}{2} C_1^a C_1^a, \quad (\text{B.3})$$

where the covariantized C_1^a is defined as in (3.16). After this, we consider gauge variation of \mathcal{L}_{2K} under gauge transformations (B.1),(B.2). In the gauge variation, we find unwanted terms proportional to the shifted curvatures \hat{R}^{abce} , \hat{R}^{ab} , \hat{R} which cannot be cancelled by modification of gauge transformations of the field entering our field content (3.1). Our observation is that those unwanted terms can be cancelled by adding to \mathcal{L}_{2K} the following contribution

$$e^{-1} \mathcal{L}_{2R} = -\frac{1}{2} \hat{R}^{cabe} \phi_0^{ab} \phi_0^{ce} + \frac{1}{2} \hat{R}^{ab} \phi_0^{ac} \phi_0^{cb} - \frac{1}{2} \hat{R}^{ab} \phi_0^{ab} \phi_0^{cc} + \left(\frac{1}{8} \phi_0^{aa} \phi_0^{bb} - \frac{1}{4} \phi_0^{ab} \phi_0^{ab} \right) \hat{R}. \quad (\text{B.4})$$

This is to say that variation of $\mathcal{L}_2 = \mathcal{L}_K + \mathcal{L}_{2R}$ under gauge transformations (B.1),(B.2) takes the form

$$\delta_{\xi_{-1}} \mathcal{L}_2 = \delta_{\phi_0^{ab}, \xi_{-1}} \mathcal{L}_2 + \delta_{b^a, \xi_{-1}} \mathcal{L}_2, \quad (\text{B.5})$$

$$\delta_{\phi_0^{ab}, \xi_{-1}} \mathcal{L}_2 = \delta_{\phi_0^{ab}, \xi_{-1}} \mathcal{L}_2|_{\hat{G}} + \delta_{\phi_0^{ab}, \xi_{-1}} \mathcal{L}_2|_F, \quad (\text{B.6})$$

$$e^{-1} \delta_{\phi_0^{ab}, \xi_{-1}} \mathcal{L}_2|_{\hat{G}} = \hat{G}^{ab} \mathcal{L}_{\xi_{-1}} \phi_0^{ab}, \quad (\text{B.7})$$

$$\begin{aligned} e^{-1} \delta_{\phi_0^{ab}, \xi_{-1}} \mathcal{L}_2|_F &= -2q F^{ac} \mathcal{F}^{cb}(\xi_{-1}) \phi_0^{ab} + 2q F^{ab} \xi_{-1}^b (\mathcal{D} \phi_0)^a - q F^{ab} \xi_{-1}^b \mathcal{D}^a \phi_0^{cc} \\ &+ 2q \mathcal{D}^a F^{ab} \xi_{-1}^c \phi_0^{bc} + q \mathcal{D}^a F^{ab} \xi_{-1}^b \phi_0^{cc}, \end{aligned} \quad (\text{B.8})$$

$$e^{-1} \delta_{b^a, \xi_{-1}} \mathcal{L}_2 = 2q \xi_{-1}^b \phi_0^{ab} (\mathcal{D}^c \phi_0^{ac} - \mathcal{D}^a \phi_0^{cc})$$

$$- 2q\mathcal{D}^a\xi^b(\phi^2)^{ab} - \frac{q}{2}\phi_0^{aa}\phi_0^{bb}\mathcal{D}\xi + \frac{3q}{2}\phi_0^{ab}\phi_0^{ab}\mathcal{D}\xi + q\mathcal{D}^a\xi^b\phi_0^{ab}\phi_0^{cc}, \quad (\text{B.9})$$

where Lie derivative $\mathcal{L}_{\xi_{-1}}\phi_0^{ab}$ entering variation (B.7) is defined in (3.44). From these relations, we see that the remaining terms involving the shifted curvatures are proportional to \widehat{G}^{ab} (B.7). Such terms can easily be cancelled by modifying gauge transformation of the field ϕ_2^{ab} ,

$$\delta'_{\xi_{-1}}\phi_2^{ab} = \mathcal{L}_{\xi_{-1}}\phi_0^{ab}. \quad (\text{B.10})$$

Namely, it is easy to see that variation of \mathcal{L}_1 (3.5) under gauge transformation of ϕ_2^{ab} given in (B.10) cancels variation in (B.7).

Step 3. We now consider variation of contributions to the Lagrangian denoted by \mathcal{L}_3 (3.7) and \mathcal{L}_5 (3.9) under gauge transformations given in (B.1),(B.2). Using notation $\delta_{b^a,\xi_{-1}}\mathcal{L}_3$ and $\delta_{b^a,\xi_{-1}}\mathcal{L}_5$ for the respective variations of \mathcal{L}_3 and \mathcal{L}_5 under ξ_{-1}^a gauge transformation of the field b^a , and notation $\delta_{\phi_0^{ab},\xi_{-1}}\mathcal{L}_5$ for variation of \mathcal{L}_5 under ξ_{-1}^a gauge transformation of the field ϕ_0^{ab} we find

$$e^{-1}(\delta_{b^a,\xi_{-1}}\mathcal{L}_3 + \delta_{\phi_0^{ab},\xi_{-1}}\mathcal{L}_5) = (\phi_1^a\xi_{-1}^b + \phi_1^b\xi_{-1}^a - \frac{1}{2}\eta^{ab}\phi_1^c\xi_{-1}^c)\widehat{G}^{ab}, \quad (\text{B.11})$$

$$e^{-1}\delta_{b^a,\xi_{-1}}\mathcal{L}_5 = \phi_1^a(\phi_0^{ab}\xi_{-1}^b + \frac{1}{4}\phi_0^{bb}\xi_{-1}^a + \frac{u}{2}\xi_{-1}^a\phi_0) + \frac{1}{4}\phi_0^{bb}\xi_{-1}^a + \frac{u}{2}\xi_{-1}^a\phi_0. \quad (\text{B.12})$$

From these relations, we see that variation proportional to \widehat{G}^{ab} (B.11) can be cancelled by modifying ξ_{-1}^a gauge transformation of the field ϕ_2^{ab}

$$\delta''_{\xi_{-1}}\phi_2^{ab} = \phi_1^a\xi_{-1}^b + \phi_1^b\xi_{-1}^a - \frac{1}{2}\eta^{ab}\phi_1^c\xi_{-1}^c. \quad (\text{B.13})$$

Namely, it is easy to see that variation of \mathcal{L}_1 (3.5) under gauge transformation of ϕ_2^{ab} (B.13) cancels variation in (B.11). Collecting results in (B.10) and (B.13), we find complete ξ_{-1}^a gauge transformation of the field ϕ_2^{ab} ,

$$\delta_{\xi_{-1}}\phi_2^{ab} = \mathcal{L}_{\xi_{-1}}\phi_0^{ab} + \phi_1^a\xi_{-1}^b + \phi_1^b\xi_{-1}^a - \frac{1}{2}\eta^{ab}\phi_1^c\xi_{-1}^c. \quad (\text{B.14})$$

Step 4. Using notation $\delta_{\phi_2^{ab},\xi_{-1}}\mathcal{L}_6$ for variation of \mathcal{L}_6 under ξ_{-1}^a gauge transformation of field ϕ_2^{ab} (B.14), we find

$$\delta_{\phi_2^{ab},\xi_{-1}}\mathcal{L}_6 = \delta_{\phi_2^{ab},\xi_{-1}}\mathcal{L}_6|_{\phi_0^{ab}\phi_0^{ce},\phi_0^{ab}\phi_0} + \delta_{\phi_2^{ab},\xi_{-1}}\mathcal{L}_6|_{\phi_0^{ab}\phi_1^c,\phi_1^a\phi_0}, \quad (\text{B.15})$$

$$\begin{aligned} e^{-1}\delta_{\phi_2^{ab},\xi_{-1}}\mathcal{L}_6|_{\phi_0^{ab}\phi_0^{ab},\phi_0^{ab}\phi_0} &\equiv -(\phi_0^2)^{ab}\mathcal{D}^a\xi_{-1}^b + \phi_0^{ab}\phi_0^{cc}\mathcal{D}^a\xi_{-1}^b + \frac{1}{4}\phi_0^{ab}\phi_0^{ab}\mathcal{D}\xi_{-1} \\ &\quad - \frac{1}{4}\phi_0^{aa}\phi_0^{bb}\mathcal{D}\xi_{-1} + \frac{u}{2}\phi_0(\xi\mathcal{D})\phi_0^{cc} + u\phi_0^{ab}\phi_0\mathcal{D}^a\xi_{-1}^b, \end{aligned} \quad (\text{B.16})$$

$$e^{-1}\delta_{\phi_2^{ab},\xi_{-1}}\mathcal{L}_6|_{\phi_0^{ab}\phi_1^c,\phi_1^a\phi_0} \equiv -\phi_1^a(\phi_0^{ab}\xi_{-1}^b + \frac{1}{4}\phi_0^{bb}\xi_{-1}^a + \frac{u}{2}\phi_0\xi_{-1}^a). \quad (\text{B.17})$$

Comparing (B.12) and (B.17), we find the cancellation

$$\delta_{b^a,\xi_{-1}}\mathcal{L}_5 + \delta_{\phi_2^{ab},\xi_{-1}}\mathcal{L}_6|_{\phi_0^{ab}\phi_1^c,\phi_1^a\phi_0} = 0. \quad (\text{B.18})$$

We proceed to the next step of our procedure with noticing that variations that remain to be cancelled are given in (B.8), (B.9), and (B.16).

Step 5. We now consider F^{ab} depending variation given in (B.8). This variation can be cancelled by adding new contributions to Lagrangian and modifying ξ_{-1}^a gauge transformations of the field ϕ_1^a . Note that, in flat conformal gravity, the field ϕ_1^a is not transformed under ξ_{-1}^a gauge transformations (see (2.56)). This is to say that, in interacting conformal gravity, we consider the following new contributions to Lagrangian and ξ_{-1}^a gauge transformation of the field ϕ_1^a :

$$e^{-1}\mathcal{L}_7 = c_1 F^{ac} F^{cb} \phi_0^{ab} + c_2 F^{ab} F^{ab} \phi_0^{cc} + c_3 F^{ab} F^{ab} \phi_0, \quad (\text{B.19})$$

$$\delta_{\xi_{-1}} \phi_1^a = f_1 \phi_0^{ab} \xi_{-1}^b + f_2 F^{ab} \xi_{-1}^b + f_3 \phi_0 \xi_{-1}^a + f_4 \phi_0^{cc} \xi_{-1}^a, \quad (\text{B.20})$$

where coefficients $c_{1,2,3}$ and $f_{1,2,3,4}$ remain to be determined. To this end we compute variations of \mathcal{L}_3 (3.7) and \mathcal{L}_5 (3.9) under ξ_{-1}^a gauge transformation of the field ϕ_1^a (B.20),

$$\begin{aligned} e^{-1} \delta_{\phi_1^a, \xi_{-1}} \mathcal{L}_3 &= f_2 \mathcal{D}^a F^{ab} F^{bc} \xi_{-1}^c + f_1 \mathcal{D}^a F^{ab} \phi_0^{bc} \xi_{-1}^c \\ &+ f_3 \mathcal{D}^a F^{ab} \xi_{-1}^b \phi_0 + f_4 \mathcal{D}^a F^{ab} \xi_{-1}^b \phi_0^{cc}, \end{aligned} \quad (\text{B.21})$$

$$\delta_{\phi_1^a, \xi_{-1}} \mathcal{L}_5 = \delta_{\phi_1^a, \xi_{-1}} \mathcal{L}_5|_F + \delta_{\phi_1^a, \xi_{-1}} \mathcal{L}_5|_{\phi_0^{ab}, \phi_0}, \quad (\text{B.22})$$

$$e^{-1} \delta_{\phi_1^a, \xi_{-1}} \mathcal{L}_5|_F \equiv f_2 F^{ab} \xi_{-1}^b (\mathcal{D} \phi_0)^a - f_2 F^{ab} \xi_{-1}^b \mathcal{D}^a \phi_0^{cc} - f_2 u F^{ab} \xi_{-1}^b \mathcal{D}^a \phi_0, \quad (\text{B.23})$$

$$\begin{aligned} e^{-1} \delta_{\phi_1^a, \xi_{-1}} \mathcal{L}_5|_{\phi_0^{ab}, \phi_0} &\equiv f_1 \phi_0^{ab} \xi_{-1}^b (\mathcal{D}^c \phi_0^{ac} - \mathcal{D}^a \phi_0^{cc}) - f_1 u \phi_0^{ab} \xi_{-1}^b \mathcal{D}^a \phi_0 \\ &+ f_3 \phi_0 \xi_{-1}^a (\mathcal{D} \phi_0)^a - f_3 \phi_0 \xi_{-1}^a \mathcal{D}^a \phi_0^{cc} + \frac{1}{2} f_3 u \phi_0^2 \mathcal{D} \xi_{-1} \\ &+ f_4 \text{ terms}. \end{aligned} \quad (\text{B.24})$$

Also, computing variation of \mathcal{L}_7 (B.19) under gauge transformations (B.1), (B.2), we find

$$\begin{aligned} e^{-1} \delta_{\xi_{-1}} \mathcal{L}_7 &= -2c_1 \mathcal{D}^a F^{ab} F^{bc} \xi_{-1}^c + (2c_2 - \frac{c_1}{2}) F^2 \mathcal{D} \xi_{-1} - 2c_1 F^{ac} \mathcal{F}^{cb}(\xi_{-1}) \phi_0^{ab} \\ &+ 4c_2 \mathcal{D}^a F^{ab} \xi_{-1}^b \phi_0^{cc} + 4c_2 F^{ab} \xi_{-1}^b \mathcal{D}^a \phi_0^{cc} \\ &+ 4c_3 \mathcal{D}^a F^{ab} \xi_{-1}^b \phi_0 + 4c_3 F^{ab} \xi_{-1}^b \mathcal{D}^a \phi_0. \end{aligned} \quad (\text{B.25})$$

Requiring the F^{ab} depending terms to cancel gives the equations

$$\delta_{\phi_0^{ab}, \xi_{-1}} \mathcal{L}_2|_F + \delta_{\phi_1^a, \xi_{-1}} \mathcal{L}_3 + \delta_{\phi_1^a, \xi_{-1}} \mathcal{L}_5|_F + \delta_{\xi_{-1}} \mathcal{L}_7 = 0, \quad (\text{B.26})$$

which allow us to fix all coefficients in (B.19), (B.20),

$$c_1 = -\frac{1}{4}, \quad c_2 = -\frac{1}{16}, \quad c_3 = -\frac{u}{8}, \quad f_1 = -\frac{1}{2}, \quad f_2 = -\frac{1}{2}, \quad f_3 = \frac{u}{2}, \quad f_4 = 0. \quad (\text{B.27})$$

Using (B.27), we note the relation $f_1 u + f_3 = 0$, which allows us to represent (B.24) as (up to total derivative)

$$e^{-1} \delta_{\phi_1^a, \xi_{-1}} \mathcal{L}_5|_{\phi_0^{ab}, \phi_0} = f_1 \phi_0^{ab} \xi_{-1}^b (\mathcal{D}^c \phi_0^{ac} - \mathcal{D}^a \phi_0^{cc}) + f_1 u \phi_0^{ab} \phi_0 \mathcal{D}^a \xi_{-1}^b$$

$$- f_3 \phi_0 \xi_{-1}^a \mathcal{D}^a \phi_0^{cc} + \frac{1}{2} f_3 u \phi_0^2 \mathcal{D} \xi_{-1}. \quad (\text{B.28})$$

Step 6. Variations that remain to be cancelled are given in (B.9),(B.16),(B.28). All these variations involve terms of the second order in the fields ϕ_0^{ab} and ϕ_0 . Note that the variation of \mathcal{L}_4 (3.8) under ξ_{-1}^a gauge transformation also gives terms of the second order in the field ϕ_0 ,

$$e^{-1} \delta_{\xi_{-1}} \mathcal{L}_4 = -\frac{1}{4} \phi_0^2 \mathcal{D}^c \xi_{-1}^c. \quad (\text{B.29})$$

We note that variations (B.9),(B.16),(B.28) and (B.29) can be cancelled by adding new contributions to Lagrangian without any additional modification of ξ_{-1}^a gauge transformations of the fields. This is to say that we consider the following new contributions to Lagrangian:

$$\begin{aligned} e^{-1} \mathcal{L}_8 &= p_1 \phi_0^{ab} \phi_0^{bc} \phi_0^{ca} + p_2 \phi_0^{ab} \phi_0^{ab} \phi_0^{cc} + p_3 (\phi_0^{aa})^3 \\ &+ p_4 \phi_0^{ab} \phi_0^{ab} \phi_0 + p_5 \phi_0^{aa} \phi_0^2 + p_6 \phi_0^3 + p_7 (\phi_0^{aa})^2 \phi_0. \end{aligned} \quad (\text{B.30})$$

Computing

$$\begin{aligned} e^{-1} \delta_{\xi_{-1}} \mathcal{L}_8 &= 6p_1 (\phi_0^2)^{ab} \mathcal{D}^a \xi_{-1}^b + 4p_2 \phi_0^{ab} \phi_0^{cc} \mathcal{D}^a \xi_{-1}^b + 2p_2 (\phi_0^2)^{aa} \mathcal{D} \xi_{-1} + 6p_3 \phi_0^{aa} \phi_0^{bb} \mathcal{D} \xi_{-1} \\ &+ 4p_4 \phi_0^{ab} \phi_0 \mathcal{D}^a \xi_{-1}^b + 2p_5 \phi_0^2 \mathcal{D} \xi_{-1} + 4p_7 \phi_0^{aa} \phi_0 \mathcal{D} \xi_{-1}, \end{aligned} \quad (\text{B.31})$$

and requiring

$$\delta_{b^a, \xi_{-1}} \mathcal{L}_2 + \delta_{\xi_{-1}} \mathcal{L}_4 + \delta_{\phi_1^a, \xi_{-1}} \mathcal{L}_5|_{\phi_0^{ab}, \phi_0} + \delta_{\phi_2^{ab}, \xi_{-1}} \mathcal{L}_6|_{\phi_0^{ab} \phi_0^{ce}, \phi_0^{ab} \phi_0} + \delta_{\xi_{-1}} \mathcal{L}_8 = 0, \quad (\text{B.32})$$

we get

$$p_1 = \frac{1}{4}, \quad p_2 = -\frac{5}{16}, \quad p_3 = \frac{1}{16}, \quad p_4 = -\frac{u}{8}, \quad p_5 = -\frac{3}{16}, \quad p_7 = 0. \quad (\text{B.33})$$

Thus we see that, with exception of the coefficient p_6 , all the remaining coefficients entering cubic potential (B.30) are fixed by ξ_{-1}^a gauge symmetries. Note also that all variations of the Lagrangian \mathcal{L} , which are proportional to the gauge transformation parameter ξ_{-1}^a , have been cancelled.

Step 7. The coefficient p_6 is fixed by considering ξ_0 gauge transformations. With exception of the field ϕ_2^{ab} , ξ_0 gauge transformations given in (3.57)-(3.62) are obtained by covariantization of the corresponding gauge transformations of free fields. Using gauge transformations in (3.57)-(3.62) we check the relations

$$\delta_{\xi_0} (\mathcal{L}_{2K} + \mathcal{L}_4 + \mathcal{L}_5) = 0, \quad \delta_{\xi_0} (\mathcal{L}_3 + \mathcal{L}_7) = 0, \quad (\text{B.34})$$

where we use the decomposition $\mathcal{L}_2 = \mathcal{L}_{2K} + \mathcal{L}_{2R}$ (see (4.4),(B.3),(B.4)). We note that \mathcal{L}_{2R} (B.4) is not invariant under ξ_0 gauge transformation

$$e^{-1} \delta_{\xi_0} \mathcal{L}_{2R} = -\frac{1}{2} \phi_0^{ab} \widehat{G}^{ab} \xi_0. \quad (\text{B.35})$$

It is easy to see that this gauge variation can be cancelled by modifying ξ_0 gauge transformation of the field ϕ_2^{ab} ,

$$\delta'_{\xi_0} \phi_2^{ab} = -\frac{1}{2} \phi_0^{ab} \xi_0. \quad (\text{B.36})$$

We now consider the remaining gauge variations to be cancelled

$$e^{-1}\delta_{\xi_0}\mathcal{L}_8 = -\frac{1}{4}\phi_0^{ab}\phi_0^{ab}\xi_0 + \frac{1}{4}\phi_0^{aa}\phi_0^{bb}\xi_0 + \frac{u}{4}\phi_0^{aa}\phi_0\xi_0 - 3\left(\frac{3}{16} + up_6\right)\phi_0^2\xi_0, \quad (\text{B.37})$$

$$e^{-1}\delta'_{\xi_0}\mathcal{L}_6 = \frac{1}{4}\phi_0^{ab}\phi_0^{ab}\xi_0 - \frac{1}{4}\phi_0^{aa}\phi_0^{bb}\xi_0 - \frac{u}{4}\phi_0^{aa}\phi_0\xi_0. \quad (\text{B.38})$$

Requiring $\delta'_{\xi_0}\mathcal{L}_6 + \delta_{\xi_0}\mathcal{L}_8 = 0$, we get $p_6 = -\frac{3}{16u}$.

Thus, with exception of ξ_1^a gauge transformations, we have checked gauge invariance of our Lagrangian with respect to all gauge transformations. The ξ_1^a gauge transformations of interacting theory (3.45)-(3.50) are simply obtained by covariantization, $\partial^a \rightarrow \mathcal{D}^a$, of the ones of flat theory (2.52)-(2.57). Doing so, we note that only the contributions to Lagrangian denoted by $\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_6$ are changed under ξ_1^a gauge transformations. Using the easily derived relations

$$\delta_{\xi_1}(\mathcal{L}_1 + \mathcal{L}_3) = 0, \quad \delta_{\xi_1}(\mathcal{L}_5 + \mathcal{L}_6) = 0, \quad (\text{B.39})$$

we see that Lagrangian (3.4) is invariant under the ξ_1^a gauge symmetries. This finishes our procedure of building the gauge invariant Lagrangian and the corresponding gauge transformations.

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